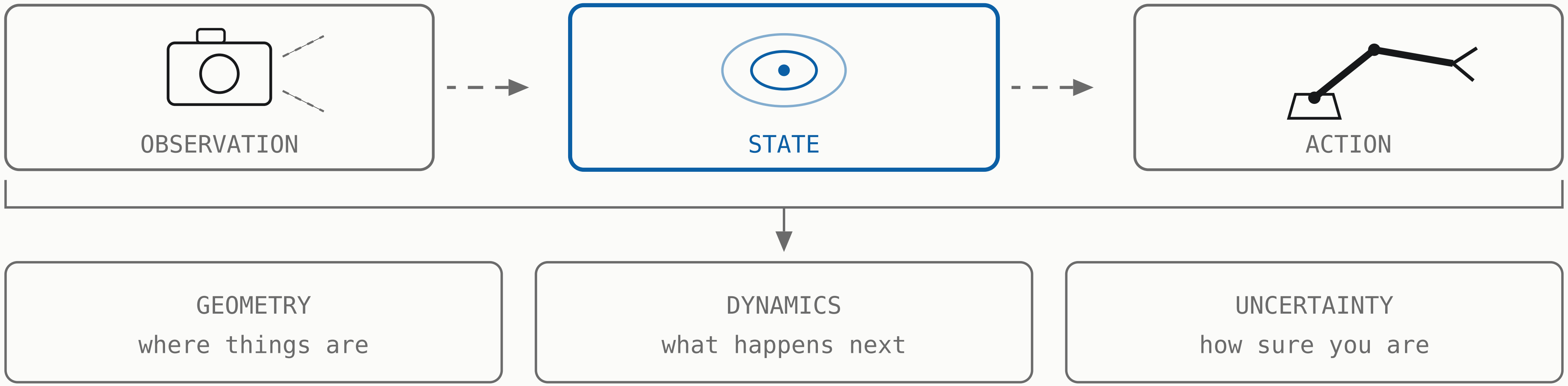
# Introduction and Linear Algebra I

## Spaces, Maps, Rank, Solutions

Mathematics for Robotics · ROB-GY 6013 · NYU Tandon

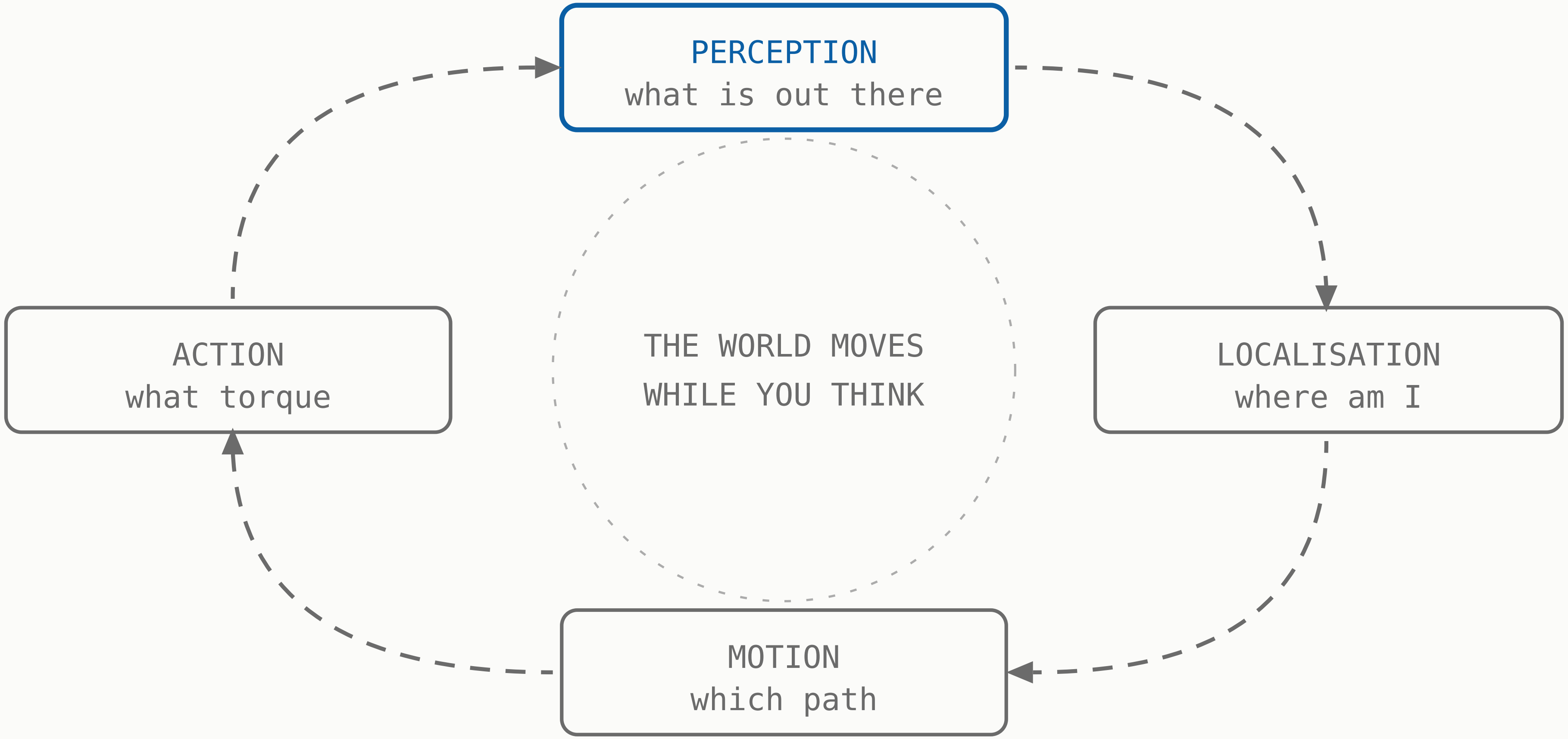
Jing Zhang · Fall 2026 · September 2

# Why Mathematics for Robotics?



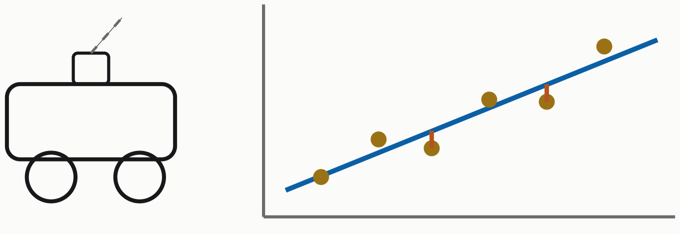

The three panels are recurring questions, not one per box above.

# Why Mathematics for Robotics?



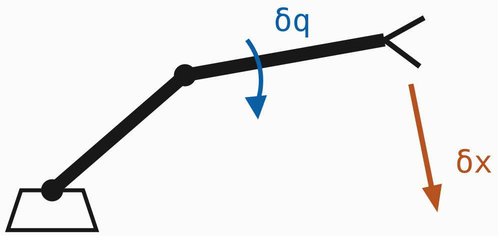

The loop closes many times a second.

# Why Mathematics for Robotics?



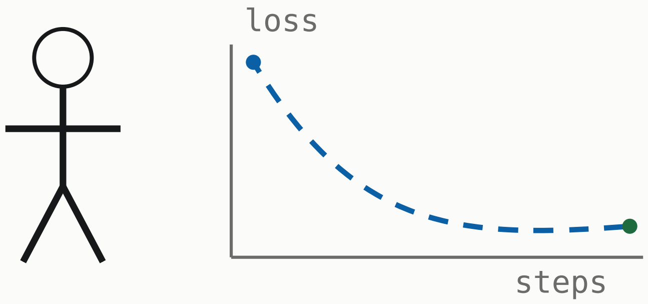
**CALIBRATION**

locally, $Ax \approx b$




**ROBOT MOTION**

$\delta x \approx J\,\delta q$




**LEARNING**

$\arg\min_\theta \mathcal{L}(\theta)$

> Different surface, same mathematical structure.

# Why Mathematics for Robotics?

Robotics changes quickly.

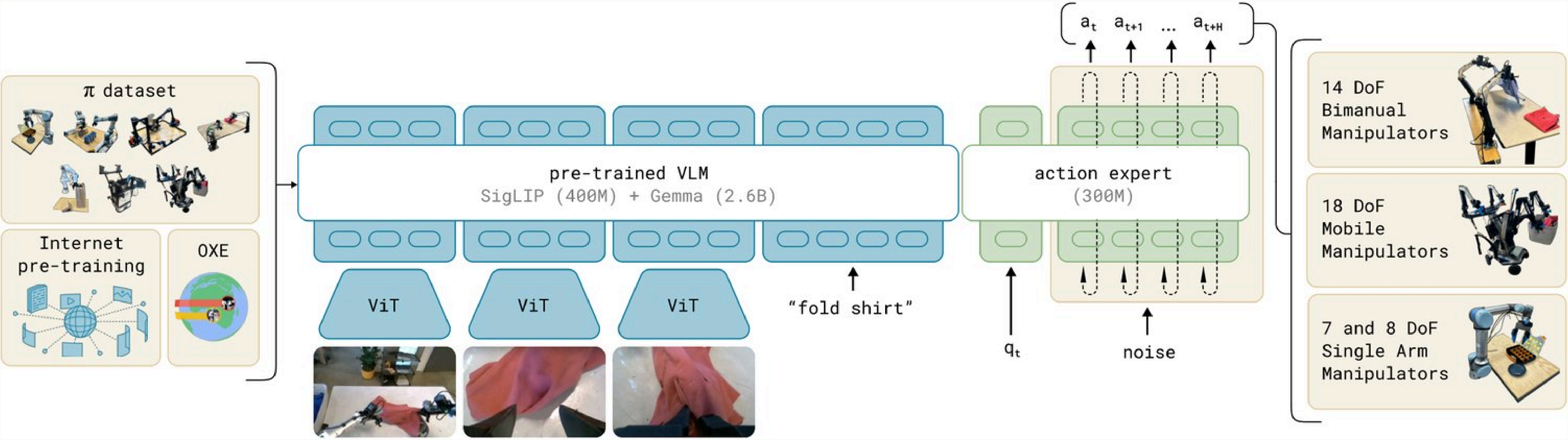
The mathematical structure changes much more slowly.

# Why Mathematics in the GenAI Era?

Google DeepMind · Advanced dexterity with Gemini Robotics 2, July 2026

opens on YouTube

# Why Mathematics in the GenAI Era?




**VLM** · attention, masks · Transformers

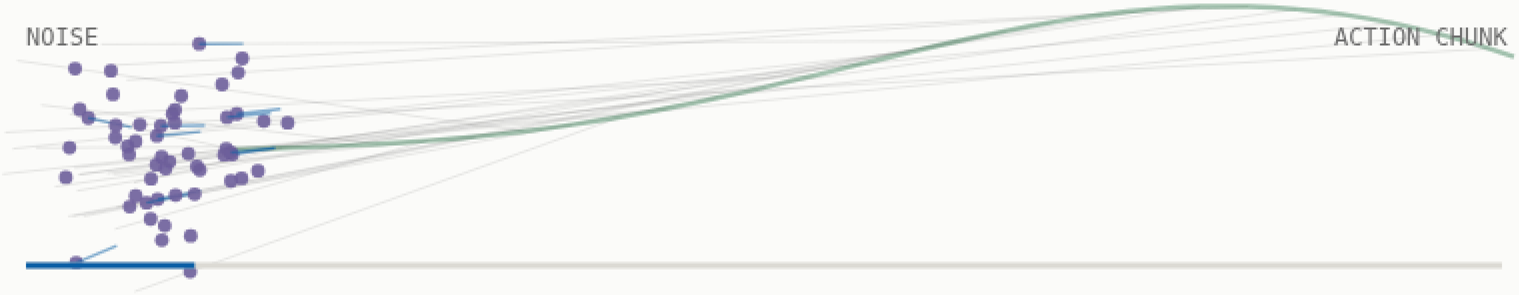
$\mathbf{q_t}$ · vector spaces · Linear Algebra I–IV

**noise → $\mathbf{a_t}$** · a vector field · ODE · Flow Matching

A modern VLA is not replacing mathematics. It is built from mathematics.

Black, K. et al. ***π₀: A Vision-Language-Action Flow Model for General Robot Control***, Figure 3. RSS, 2025. arXiv:2410.24164.

# Why Mathematics in the GenAI Era?

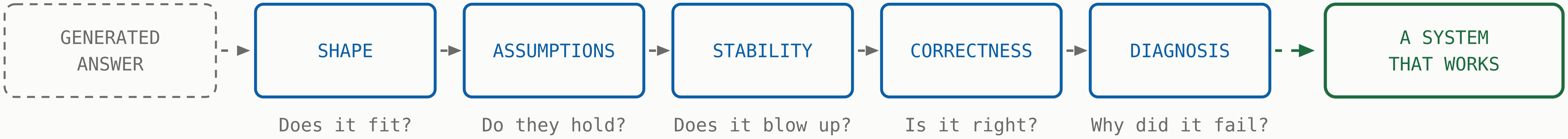



$$\boxed{\frac{dA_\tau}{d\tau} = v_\theta(A_\tau, o, \tau)} \qquad A_0 = \epsilon, \qquad \hat{a} = A_1$$

| | |
|---|---|
| $\tau$ | flow time, 0 to 1; the field changes with it |
| $v_\theta$ | the trained vector field |
| $A_0$ | the start: pure noise ε |
| $\hat{a} = A_1$ | the action chunk, read off at τ = 1 |

Black, K. et al. ***π*₀**. RSS, 2025. arXiv:2410.24164.

# Why Mathematics in the GenAI Era?




> AI can generate. You still have to judge.

# Why Mathematics in the GenAI Era?

WHAT AN ML INTERVIEW ASKS ABOUT

| | | |
|---|---|---|
| **Rank / span ●** | **Null / image ●** | **Independence ●** |
| **Inverse ●** | **Singular ●** | Orthogonality |
| Positive semidefinite | Eigenvalues | Jacobian |
| Hessian | Determinant | Dot product |

AND TO WRITE, BY HAND — M5 · M6 · M10

| | | |
|---|---|---|
| transformer end-to-end | causal, cross, self attention | flash attention |
| attention backward pass | MLP forward and backward | SGD training loop |

> “…with AI assistance completely off.”

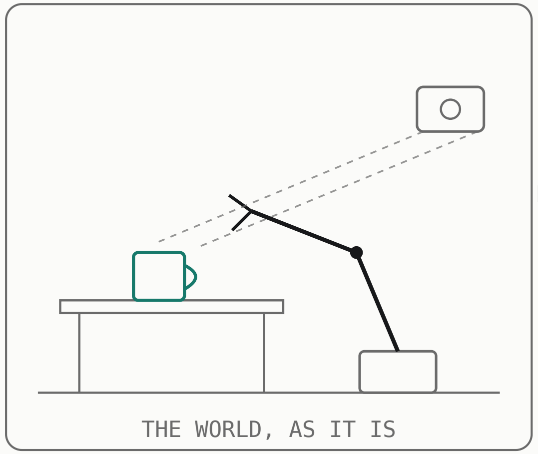

Sapora, S. ***ML Job Interviews: The Ultimate Guide***, 2026 · silviasapora.github.io/blog/ml-interviews.html
Liu, A. ***Notes on the Industry Job Search***, 2026 · alisawuffles.github.io/blog/job-search/

# The Math We Keep Coming Back To

| | | |
|---|---|---|
| x | state | Linear Algebra I · ODE · Probability I |
| A | linear map | Linear Algebra I–IV |
| f | nonlinear function | ML Math I · ODE |
| J | Jacobian | Linear Algebra IV · ML Math I–II |
| p | probability distribution | Probability I–II · Diffusion |
| L | loss | ML Math II · Diffusion · Flow Matching |

> Different problems, the same mathematical ideas.

# The Math We Keep Coming Back To




REPRESENT $\dashrightarrow$

STATE VECTOR

$$\begin{bmatrix} 0.42 \\ -1.05 \\ 0.90 \\ 0.13 \\ 2.71 \end{bmatrix}$$

JUST NUMBERS

**FOUR ROBOT QUANTITIES**

| | |
|---|---|
| $q \in \mathbb{R}^7$ | joint configuration |
| $p \in \mathbb{R}^3$ | position |
| $I \in \mathbb{R}^{3\times H\times W}$ | image |
| $a \in \mathbb{R}^{16\times 7}$ | action chunk |

Each is a vector, or an array that can be vectorized.

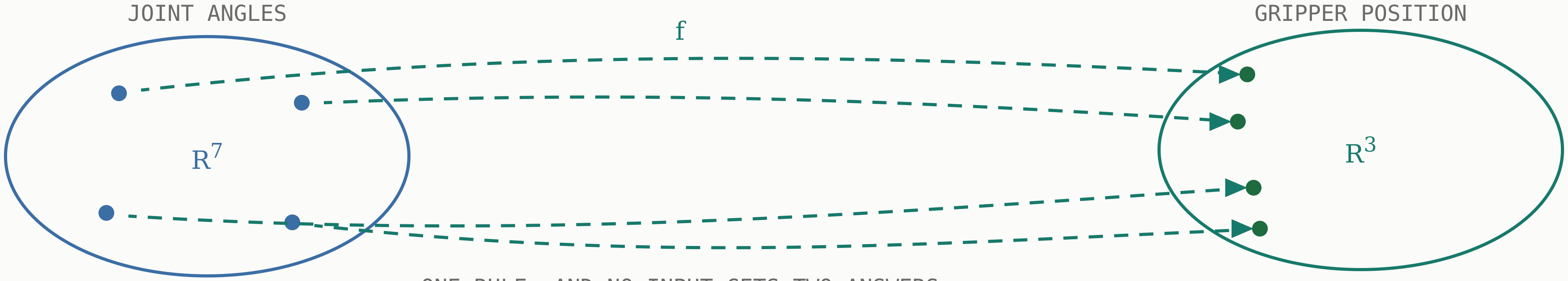
# The Math We Keep Coming Back To

**MAP** $f : \mathbb{R}^n \to \mathbb{R}^m, \qquad u \;\mapsto\; y = f(u)$

`domain` $\mathbb{R}^n$, the allowed inputs

`codomain` $\mathbb{R}^m$, the space the outputs live in

`u ↦ y` the rule that carries one to the other




ONE RULE, AND NO INPUT GETS TWO ANSWERS

Forward kinematics: $\mathbb{R}^7 \to \mathbb{R}^3$, joint angles to gripper position.

# The Math We Keep Coming Back To

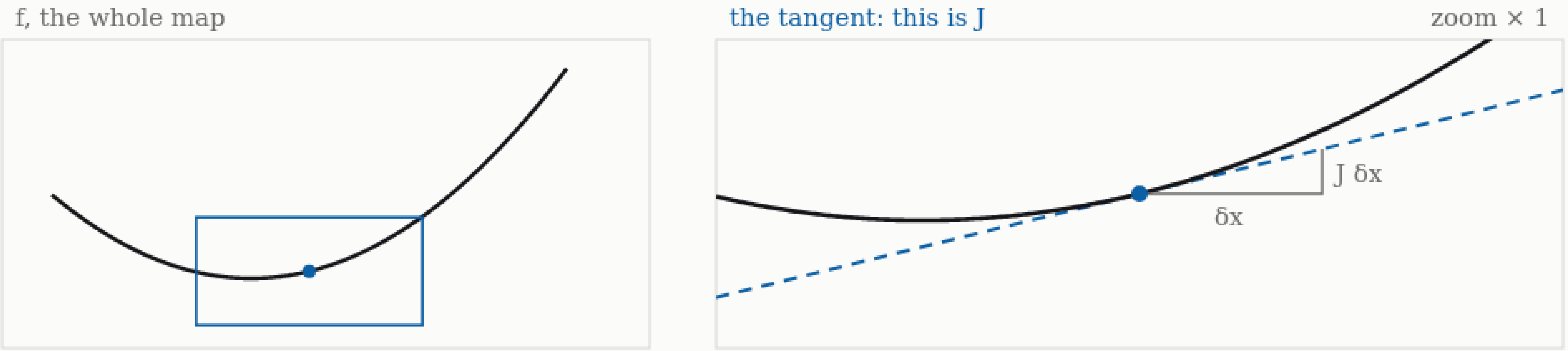
LOCAL LINEARISATION $$\boxed{f(x+\delta x) \;\approx\; f(x) + J(x)\,\delta x}$$

x the current point

δx a small change

J(x) the Jacobian, $J(x) \in \mathbb{R}^{m\times n}$

> Many nonlinear robotics problems become linear when viewed locally.

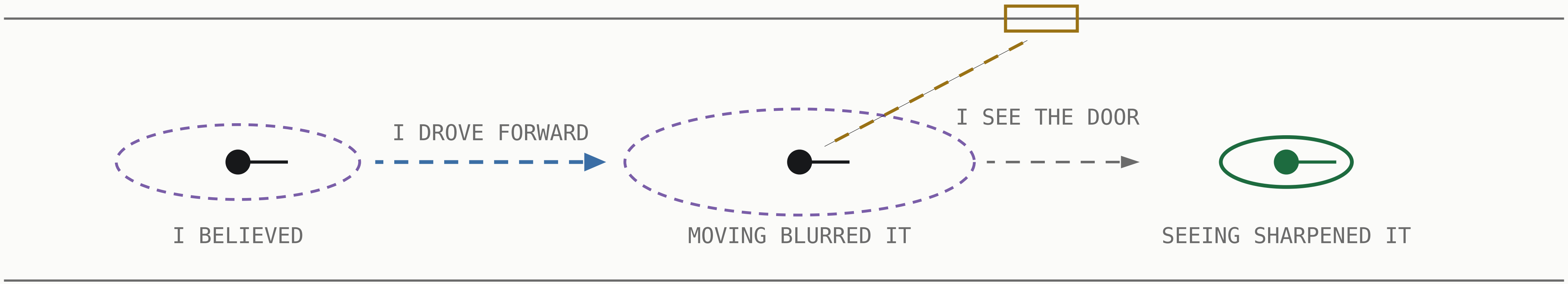



Left, the shrinking window. Right, what is inside it.

# The Math We Keep Coming Back To

**WHERE AM I?** $x_t = f(x_{t-1}, u_t) + w_t, \qquad z_t = h(x_t) + v_t$

| | |
|---|---|
| $x_t$ | the latent state, usually not observed directly |
| $u_t$, $z_t$ | the command, the measurement |
| $f$, $h$ | motion and sensor models |
| $w_t$, $v_t$ | noise |




Thrun, S., Burgard, W., Fox, D. ***Probabilistic Robotics***, Ch. 7. MIT Press, 2005.

# The Math We Keep Coming Back To

$$\hat{x} = \arg\min_x \| z - h(x) \|^2_R$$

LINEAR ALGEBRA
M1 – M4

OPTIMIZATION
M6

GEOMETRY
M3

CALCULUS
M5

PROBABILITY
M8 · M9

> $\|e\|^2_R = e^\top R^{-1} e$, with $R$ the measurement covariance.

Barfoot, T. D. ***State Estimation for Robotics***, Chapter 4. Cambridge University Press, 2017.

# Course information

THE MEETING

| | |
|---|---|
| WHEN | Wednesdays 5:00–7:30 PM |
| WHERE | 2 MetroTech Center, Room 909 |
| ONLINE | nyu.zoom.us/j/95222620311 |
| UPDATES | slides and recordings on the course page |
| QUESTIONS | Ed Discussion, linked from the course page |

THE STAFF

**Jing Zhang**

instructor · z.jing@nyu.edu

**Shichen Zhang**

teaching assistant · sz4968@nyu.edu

office hours to be announced

Everything on these four pages, plus the schedule and the reading: [jingz6676.github.io/rob6013.html](jingz6676.github.io/rob6013.html)

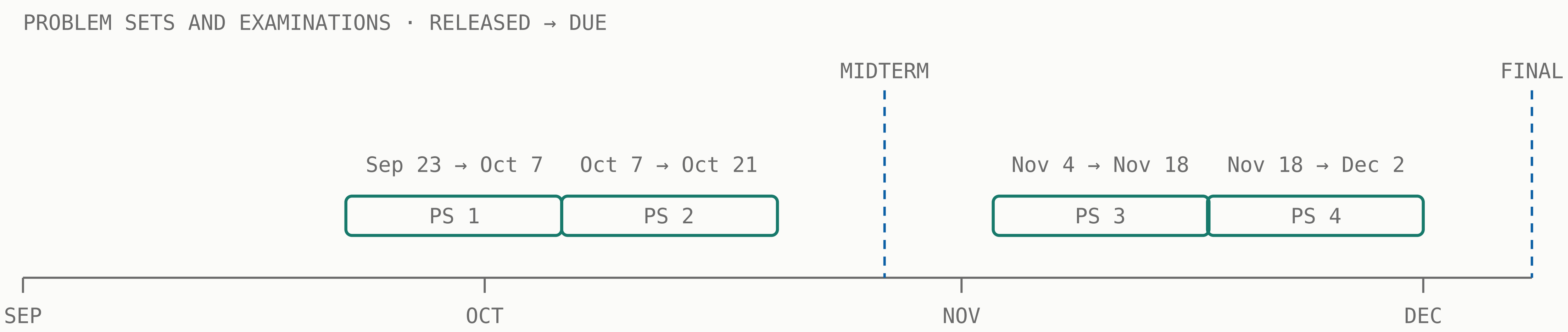
# Course information

GRADING

| 40% | 25% | 35% |
| --- | --- | --- |
| FOUR PROBLEM SETS | MIDTERM | FINAL |
| understanding, not recall | 28 October, in person | 9 December, in person |

up to +10% on the course grade, for sets typeset in LaTeX or Markdown

# Course information




> One sheet of paper, both sides, prepared by you, and a scientific calculator.
> No phone, laptop, tablet, or watch.

# Course information

POLICIES

| | |
|---|---|
| LATE WORK | 10% of the score per calendar day; closed once solutions are posted |
| COLLABORATION | discuss with anyone, then write your solution alone |
| **USING AI** | any tool to understand it, none to write your solutions |
| ACCOMMODATIONS | through the Moses Center, with no need to tell me why |
| IF IT GOES WRONG | email me and Shichen |

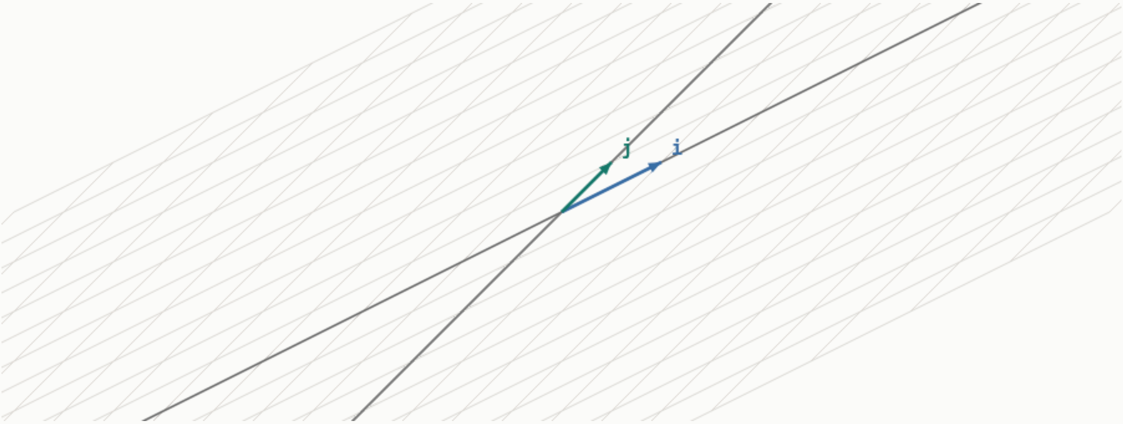
LINEAR ALGEBRA I

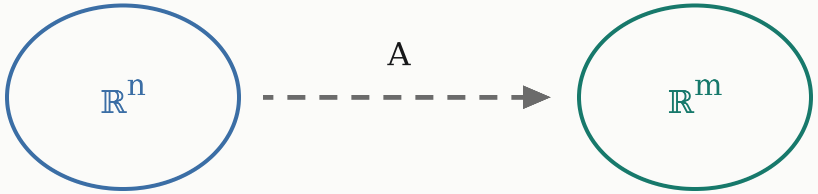
# Spaces, Maps, Rank, Solutions

# A matrix is a map

$$A \in \mathbb{R}^{m\times n}, \qquad x \in \mathbb{R}^n \longmapsto Ax \in \mathbb{R}^m$$

- `A` the map itself, m rows by n columns
- `x` the input, a column of n numbers
- `Ax` the output, a column of m numbers
- `n` input dimension: the number of columns
- `m` output dimension: the number of rows; m and n need not be equal



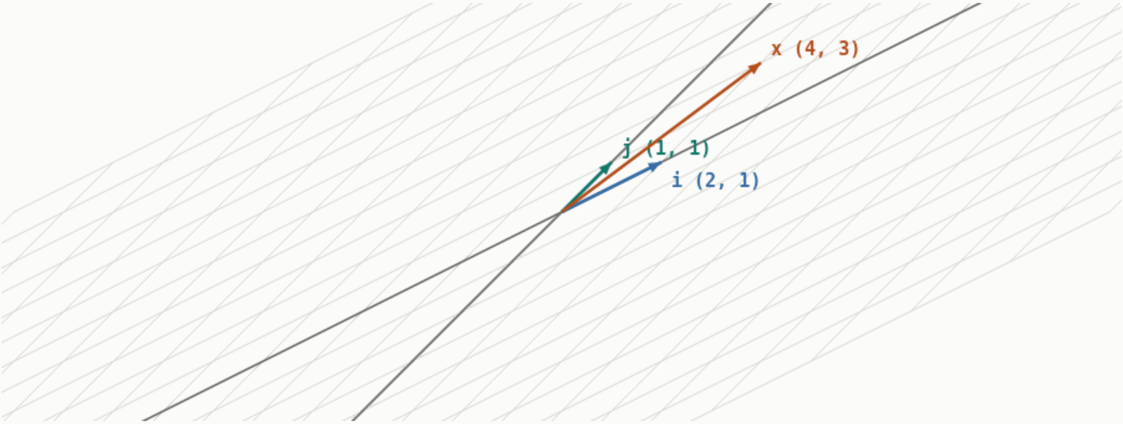

Sanderson, G. ***Essence of Linear Algebra***, ch. 3. 3Blue1Brown, 2016.

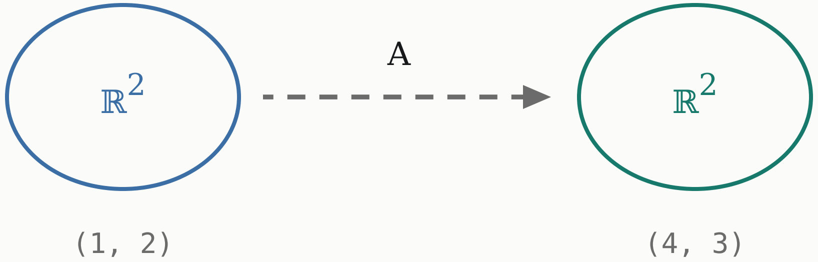
# A matrix is a map

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ the first column, where $e_1$ lands

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the second column, where $e_2$ lands



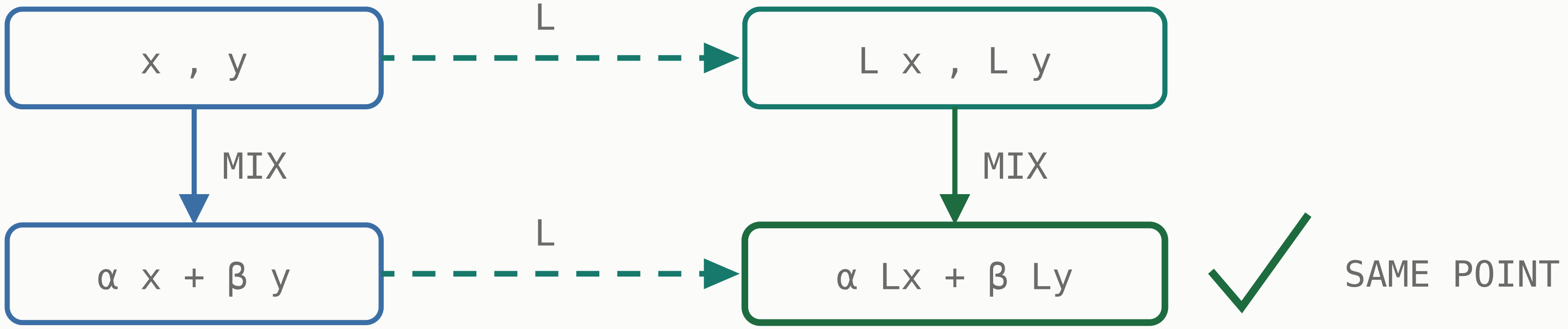
# A matrix is a map

LINEAR MAP $\boxed{L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)}$

additivity $L(x + y) = L(x) + L(y)$, the case $\alpha = \beta = 1$

homogeneity $L(\alpha x) = \alpha L(x)$, the case $\beta = 0$

superposition both at once, for all $x, y$ and all $\alpha, \beta \in \mathbb{R}$

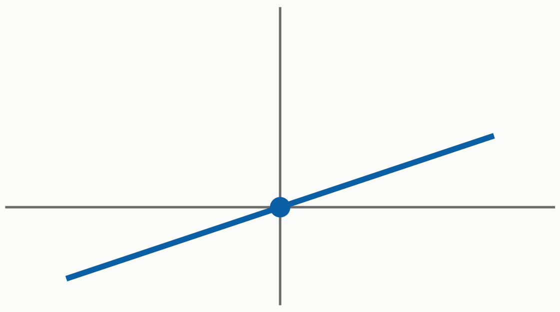



Axler, S. ***Linear Algebra Done Right***, 4th ed., Definition 3.1, p. 52. Springer, 2024. Open access.

QUESTIONS

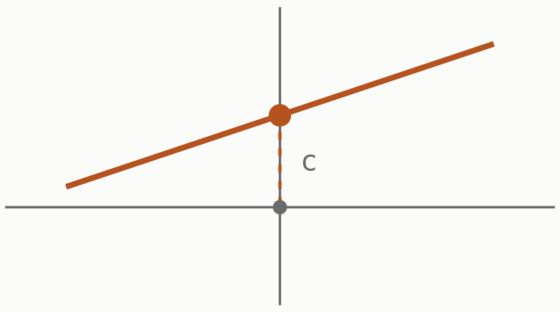
# Which of these three maps is linear?

f(x) = a x



LINEAR

f(x) = a x + c,  c ≠ 0

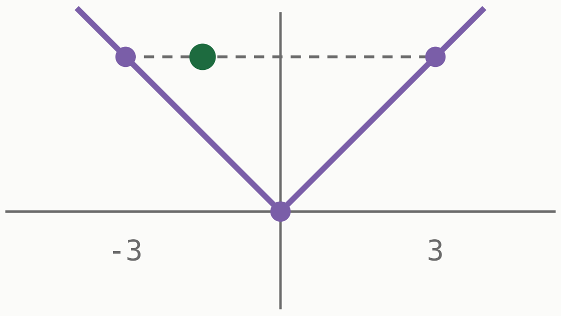



AFFINE, NOT LINEAR

a = 1,  c = 2

f(0) = 2 ≠ 0

f(x) = |x|




NOT LINEAR

f(0) = 0

f(-3) = 3,  -f(3) = -3

# A matrix is a map

$$e_j = (0, \ldots, 1, \ldots, 0)^{\mathsf{T}}$$

the 1 sits in slot j; every other slot holds 0

**COLUMN RULE** $\boxed{Ae_j = a_j}$

$a_j$ column j of A, a whole output vector

$$\underset{\text{e3}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \dashrightarrow \overset{\text{A}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}} \dashrightarrow \underset{\text{A e3}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}$$

a3

ASK FOR ONE DIRECTION, READ OFF ONE COLUMN

Axler, S. ***Linear Algebra Done Right***, 4th ed.: standard basis 2.27(a), p. 39; columns 3.50, p. 76. Springer, 2024.

# A matrix is a map

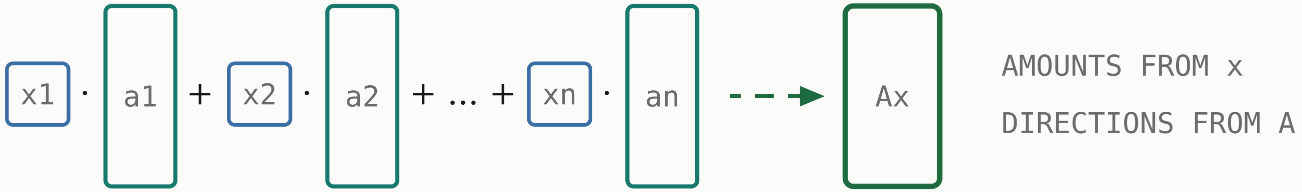
LINEAR COMBINATION OF COLUMNS

$$Ax = \sum_{j=1}^{n} x_j \,(Ae_j) = x_1 a_1 + \cdots + x_n a_n$$

$x_j$ coordinate j of x, used as an amount

$Ae_j$ column j, from the page before

$a_j$ column j of A, a whole direction




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.50 linear combination of columns, p. 76. Springer, 2024.

# A matrix is a map

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 3a_1 + 2a_2 = \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}$$

$a_1$, $a_2$ (1,0,2) and (0,1,1), the first two columns

3, 2, 0 the amounts asked of the three columns

$$3 \cdot \underset{\text{a1}}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} + 2 \cdot \underset{\text{a2}}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} + 0 \cdot \underset{\text{a3}}{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}} = \underset{\text{Ax}}{\begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix}}$$

THREE COLUMNS
THREE AMOUNTS
ONE POINT

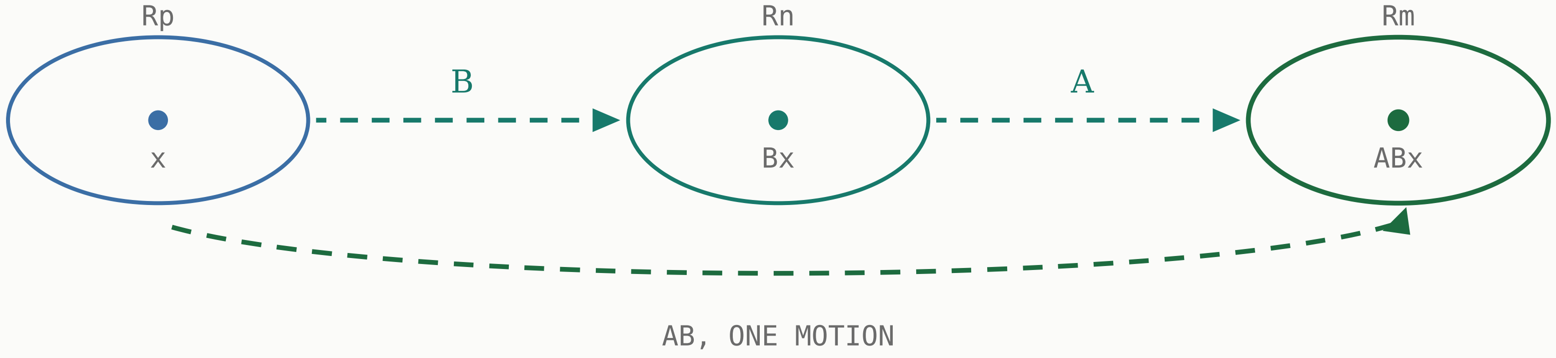
# A matrix is a map

**DO B FIRST, THEN A** $(AB)x = A(Bx)$

shapes $B \in \mathbb{R}^{n \times p}$, $A \in \mathbb{R}^{m \times n}$

Bx where B sends x first

A(Bx) where A then sends that




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.43 matrix of product of linear maps, p. 74. Springer, 2024.

# A matrix is a map

**TRANSPOSE** $(A^{\mathsf{T}})_{k,j} = A_{j,k}$

`row j` of A becomes column j of $A^{\mathsf{T}}$

`m × n` becomes n × m

**IT REVERSES A PRODUCT** $\boxed{(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}}$

`(AB)ᵀ` p × m, the shape the answer must have

`BᵀAᵀ` p × n times n × m, so n meets n

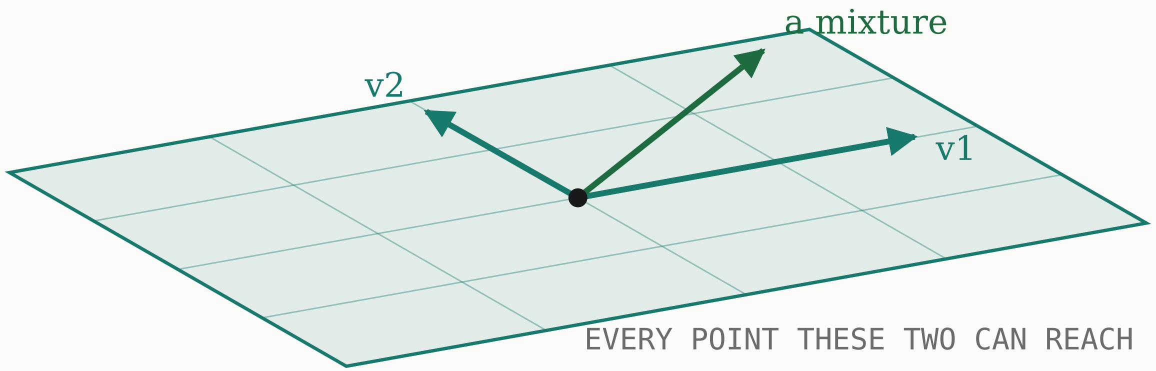
$$\overset{A}{\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \end{bmatrix}} \dashrightarrow \overset{A^{\mathsf{T}}}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$$

$$\overset{(AB)^{\mathsf{T}}}{\boxed{p\times m}} = \overset{B^{\mathsf{T}}}{\boxed{p\times n}} \cdot \overset{A^{\mathsf{T}}}{\boxed{n\times m}} \quad \checkmark \quad \text{n MEETS n}$$

$$\overset{A^{\mathsf{T}}}{\boxed{n\times m}} \times \overset{B^{\mathsf{T}}}{\boxed{p\times n}}$$

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.54 transpose, p. 77; $(AC)^t = C^tA^t$, p. 78. Springer, 2024.

# Span and column space



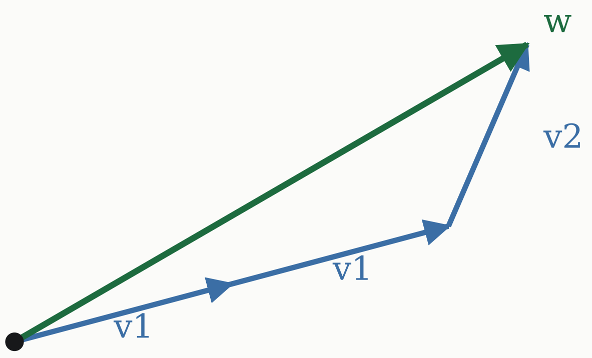
# Span and column space

**LINEAR COMBINATION** $w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$

`v_i` the directions you are handed

`c_i` any reals, negative and zero included

`w` where those weights land you




TWO STEPS OF v1

ONE STEP OF v2

w = 2 v1 + v2

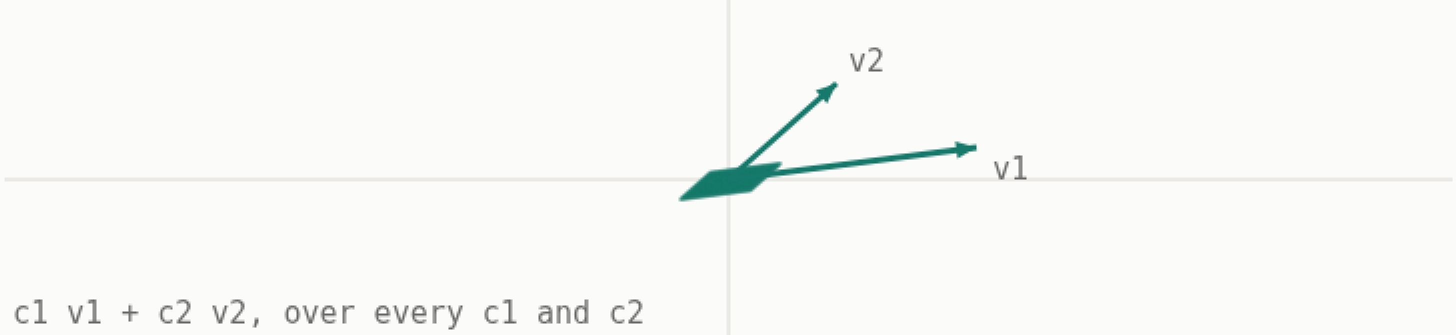
Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.2 linear combination, p. 28. Springer, 2024.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

# Span and column space

SPAN $$\operatorname{span}(v_1, \ldots, v_k) = \{c_1 v_1 + \cdots + c_k v_k \;:\; c_i \in \mathbb{R}\}$$

{ } every w the weights can reach, all at once



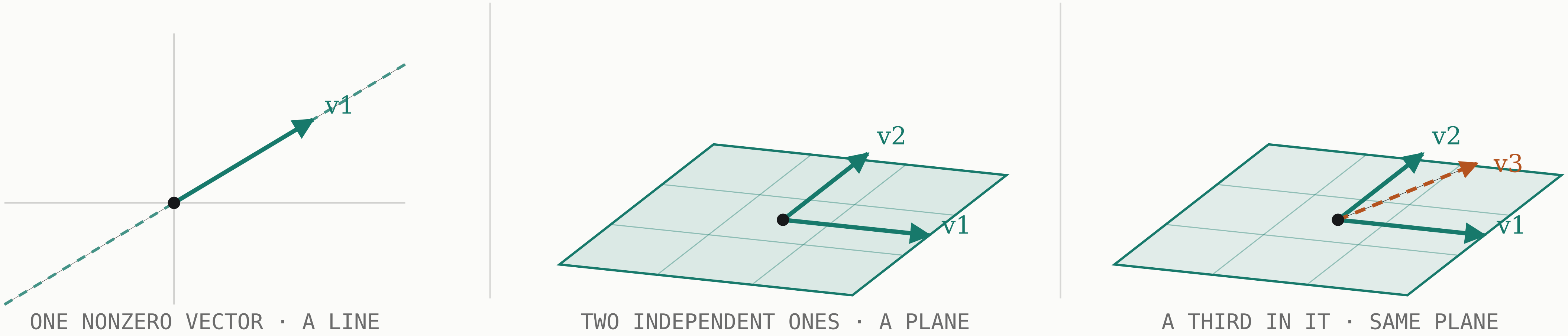

Let both weights roam: the plane fills in.

**WORKED EXAMPLE**

$v_1 = (1, 0, 2)$, $v_2 = (0, 1, 1)$: $3v_1 + 2v_2 = (3, 2, 8)$ is in the span.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.4 span, p. 29. Springer, 2024.

# Span and column space




$$\mathrm{span}(v_1) = \{c\, v_1\}$$

$$\mathrm{span}(v_1, v_2) = \{c_1 v_1 + c_2 v_2\}$$

$$\mathrm{span}(v_1, v_2, v_3) = \mathrm{span}(v_1, v_2)$$

**the conditions** zero spans only $\{0\}$; a dependent pair spans a line

**$v_3$** already a mixture of the first two

Sanderson, G. ***Essence of Linear Algebra***, Chapter 2. 3Blue1Brown, 2016.

# Span and column space

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$b \in \operatorname{span}(v_1, \ldots, v_k) \iff Vc = b \text{ has a solution}$$

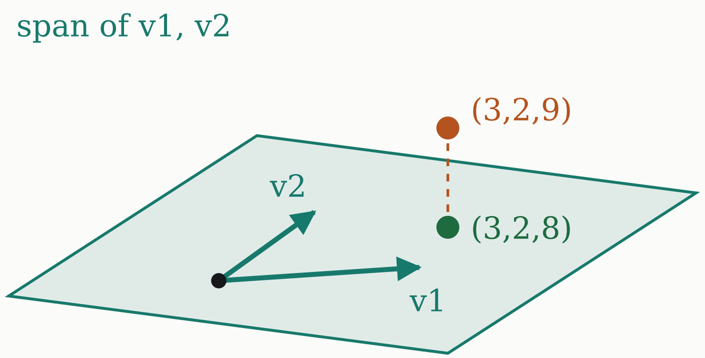
- `V` the arrows as columns
- `c` the unknown weights
- `b` the target vector

**IS (3,2,9) IN THE SPAN?**

$c_1(1,0,2) + c_2(0,1,1) = (3,2,9)$

row 1 gives $c_1 = 3$, row 2 gives $c_2 = 2$

row 3 is then forced: $2(3) + 2 = 8$, not 9



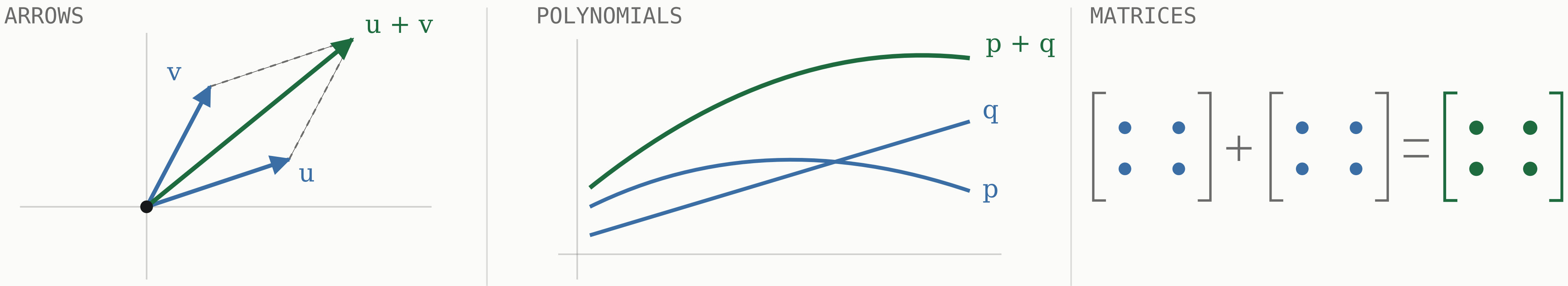

One lands, one misses.

# Span and column space

A vector space is a set equipped with vector addition and scalar multiplication satisfying the vector-space axioms.

WHAT WE USE $\boxed{u, v \in V,\ \alpha, \beta \in \mathbb{R} \implies \alpha u + \beta v \in V}$

closure the one property this course leans on all term



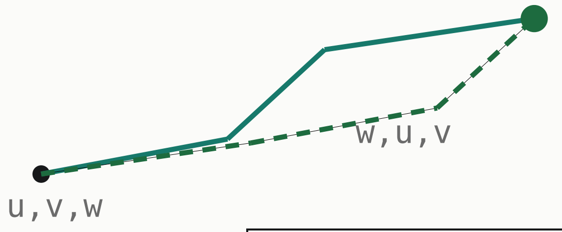

Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.20 vector space, p. 12. Springer, 2024.

# Span and column space

ADDITION

$$\boxed{\begin{array}{ll} u+v=v+u & (u+v)+w=u+(v+w) \\ v+0=v & \exists\, w:\ v+w=0 \end{array}}$$

`0 and` $w$ an origin, and an undo for every vector




ANY ORDER, SAME POINT

SCALING

$$\boxed{\begin{array}{cc} 1v=v & \alpha(\beta v)=(\alpha\beta)v \\ \alpha(u+v)=\alpha u+\alpha v & (\alpha+\beta)v=\alpha v+\beta v \end{array}}$$

`the payoff` obey all eight and every theorem here applies

Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.20 vector space, p. 12. Springer, 2024.

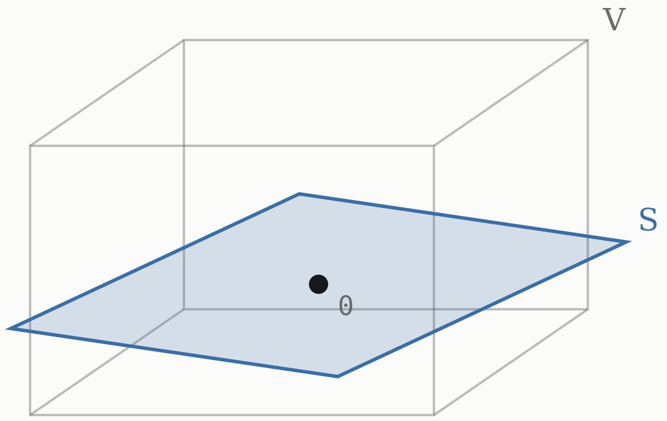
# Span and column space

A subset $S \subseteq V$ is a subspace if it is itself a vector space under the same operations.

EQUIVALENT TEST $\boxed{S \neq \varnothing \ \text{ and } \ u, v \in S, \ \alpha, \beta \in \mathbb{R} \Rightarrow \alpha u + \beta v \in S}$

`nonempty` plus closure already gives $0 \in S$: choose $\alpha = \beta = 0$

`next page` the same test, split into three checks

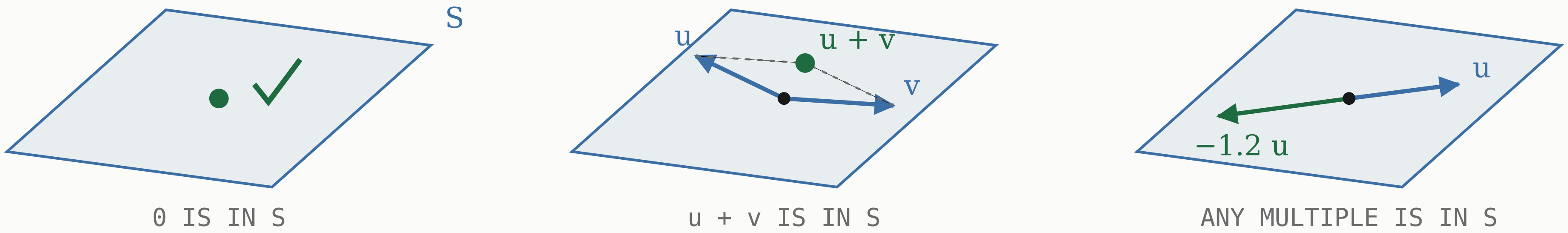



Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.33 subspace and 1.34 conditions for a subspace, p. 18. Springer, 2024.

# Span and column space

CONDITIONS FOR A SUBSPACE $0 \in S; \qquad u + v \in S; \qquad \alpha u \in S$

`0` the zero vector is a member
`u+v` any two members add to a member
`alpha u` any multiple stays in, negatives included

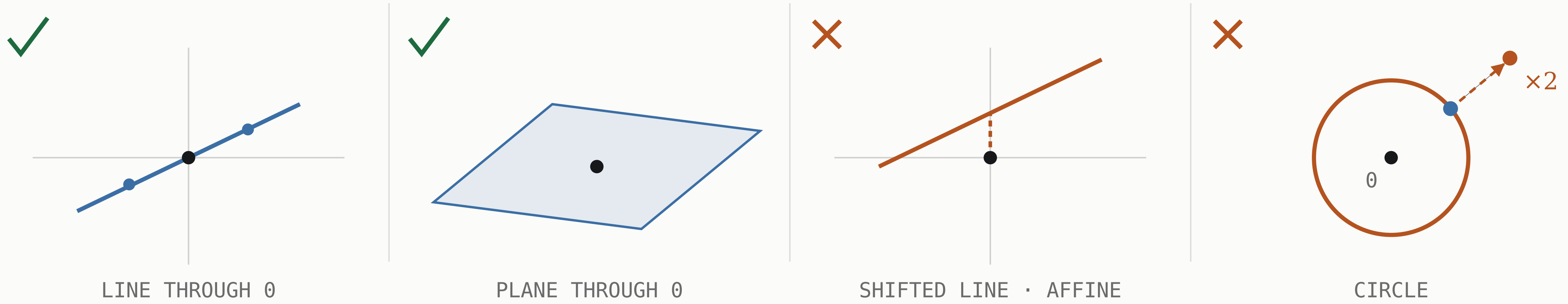



Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.34 conditions for a subspace, p. 18. Springer, 2024.

# Span and column space

FASTEST DISQUALIFIER $0 \notin S \implies S$ is not a subspace

$0 \in S$ necessary, never sufficient: closure still has to hold




Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.35 example: subspaces, p. 19. Springer, 2024.

# Span and column space

CLAIM $\boxed{v_1, \ldots, v_k \in V \implies \operatorname{span}(v_1, \ldots, v_k) \text{ is a subspace of } V}$

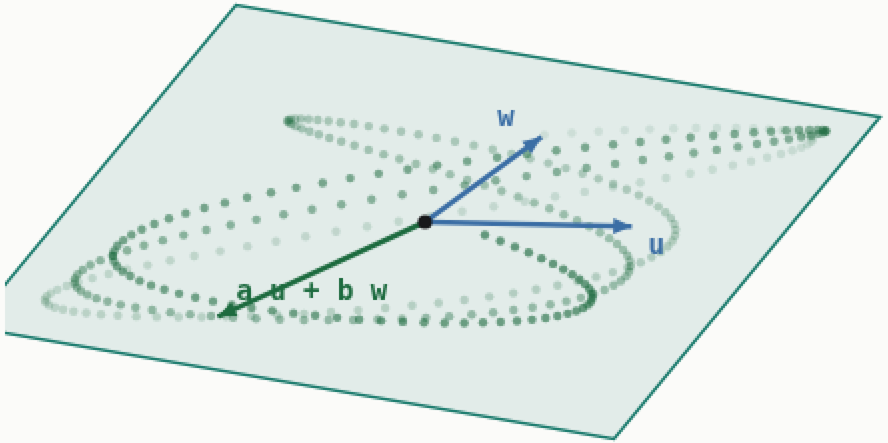
S every mixture of the list; the smallest subspace containing it

$$0 = \sum_{i=1}^{k} 0\, v_i \in S$$

every $c_i = 0$ the origin is a member

$$\alpha u + \beta w = \sum_{i=1}^{k} (\alpha c_i + \beta d_i)\, v_i \in S$$

u, w any two members: $u = \sum_i c_i v_i$, $w = \sum_i d_i v_i$

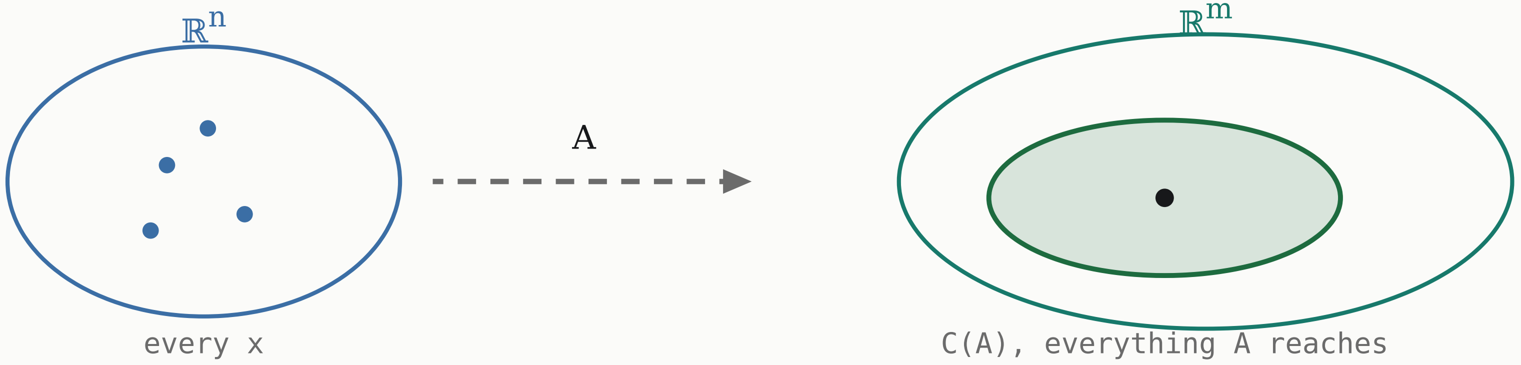



Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.6 span is the smallest containing subspace, p. 29. Springer, 2024.

# Span and column space

COLUMN SPACE $\boxed{\mathcal{C}(A) = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m}$

| | |
|---|---|
| A | any $m \times n$ matrix; x runs over the whole of $\mathbb{R}^n$ |
| $\subseteq \mathbb{R}^m$ | a subspace of the codomain; if $m = n$ the roles still differ |
| also called | the image of A, or the range of A |




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range and 3.18 the range is a subspace, p. 61. Springer, 2024.

# Span and column space

DEFINITION

$$\mathcal{C}(A) = \{Ax : x \in \mathbb{R}^n\}$$

`x` runs over every input

COLUMN VIEW

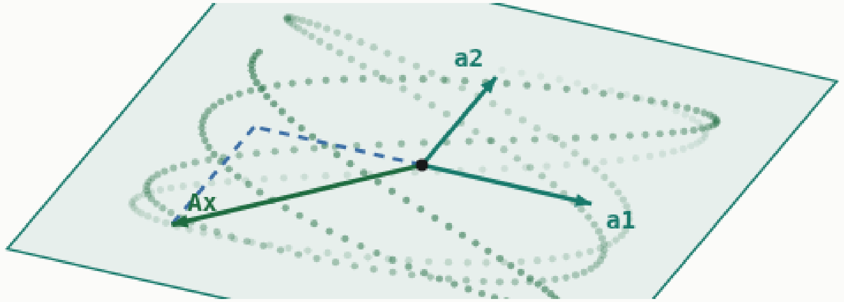
$$Ax = \sum_{j=1}^{n} x_j \, a_j$$

`aj` column j of A

$$\Downarrow$$

$$\boxed{\mathcal{C}(A) = \operatorname{span}(a_1, \ldots, a_n)}$$

`one set` reachable outputs and mixtures of the columns are the same thing




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range, p. 61; 3.50 linear combination of columns, p. 76. Springer, 2024.

# Span and column space

QUESTION $Ax = b\,?$

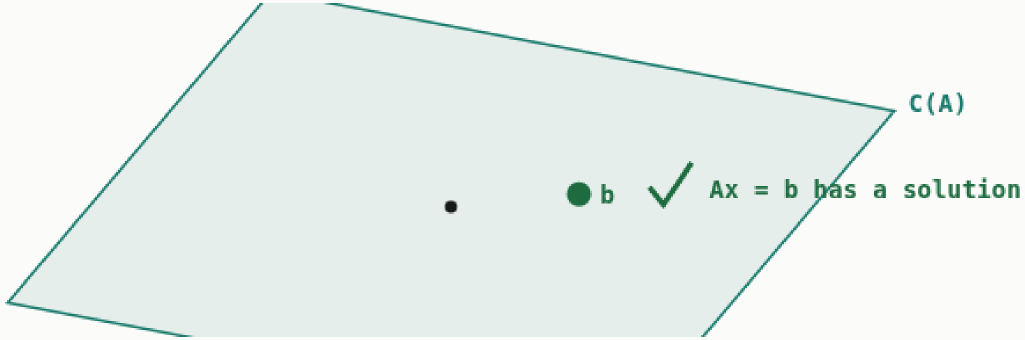
b a target on the output side

$$Ax = b \text{ has a solution} \iff \exists\, x:\ Ax = b \iff b \in \{Ax : x \in \mathbb{R}^n\}$$

∃ x the whole question is whether one input reaches b

$$\boxed{Ax = b \text{ is solvable} \iff b \in \mathcal{C}(A)}$$

nothing new the right side is the definition of C(A), read backwards




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range, p. 61. Springer, 2024.

# Span and column space

SOLVE $c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ y_3 \end{bmatrix}$

row 1 $c_1 + 0 = 3$, so $c_1 = 3$

row 2 $0 + c_2 = 2$, so $c_2 = 2$

row 3 $2c_1 + c_2 = 2(3) + 2 = 8$ must equal $y_3$

TARGET 1

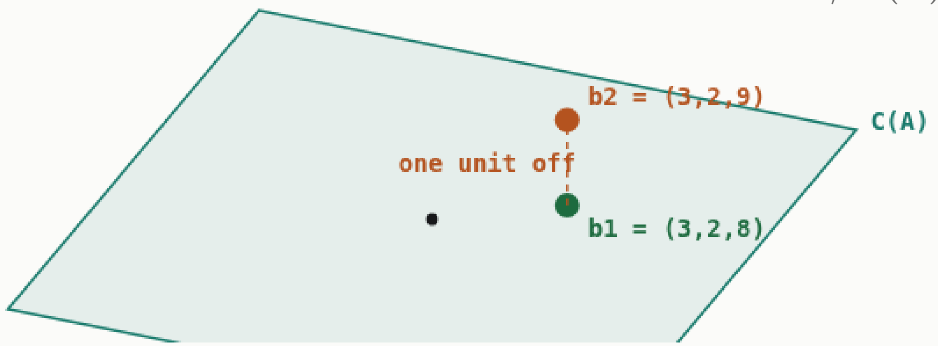
$\boxed{y_3 = 8 : \ 8 = 8 \checkmark}$

so $b_1 = 3a_1 + 2a_2 \in \mathcal{C}(A)$

TARGET 2

$\boxed{y_3 = 9 : \ 9 \neq 8 \times}$

so no $c$ works: $b_2 \notin \mathcal{C}(A)$




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range, p. 61. Springer, 2024.

# Span and column space

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$\boxed{Ax = \sum_j x_j\, a_j}$$

$x_j$ the amount taken of column j

$\Downarrow$

$$\boxed{\mathcal{C}(A) = \mathrm{span}(a_1, \ldots, a_n)}$$

span every mixture, so every reachable output

$\Downarrow$

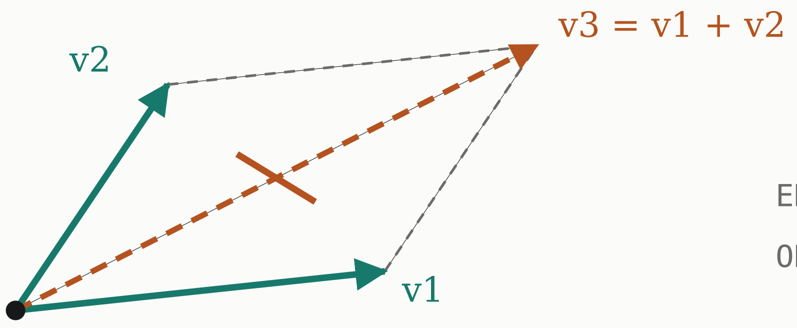
$$\boxed{Ax = b \text{ solvable} \iff b \in \mathcal{C}(A)}$$

our A $a_3 = a_1 + a_2$, so the third column adds nothing to the span

> When do different columns carry independent information?

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.15 linearly independent, p. 32. Springer, 2024.

# Null Space and Linear Independence




ENOUGH TO REACH?

OR ONE TOO MANY?

# Null Space and Linear Independence

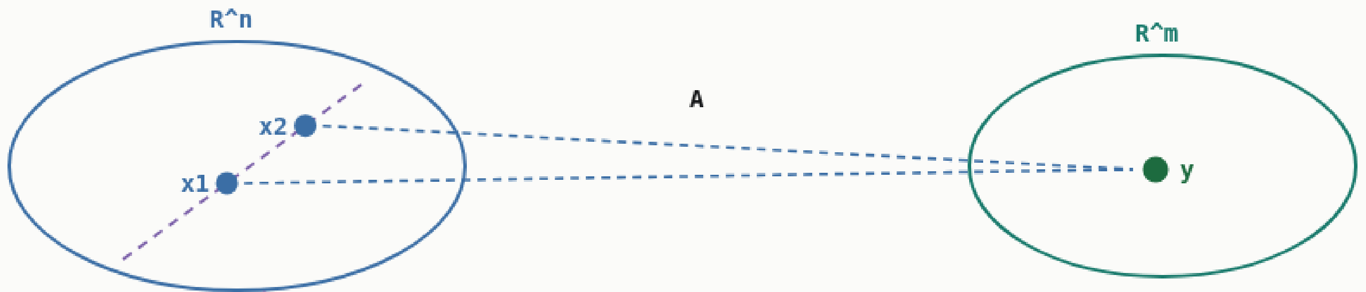
SO FAR $\mathcal{C}(A) = \text{all reachable outputs}$

settled which targets can be hit, and by which mixtures

NOW ASK $\boxed{Ax_1 = Ax_2, \qquad x_1 \neq x_2}$

one output but more than one input that reaches it

> How can two different inputs produce the same output?




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.14 injective, p. 60; 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

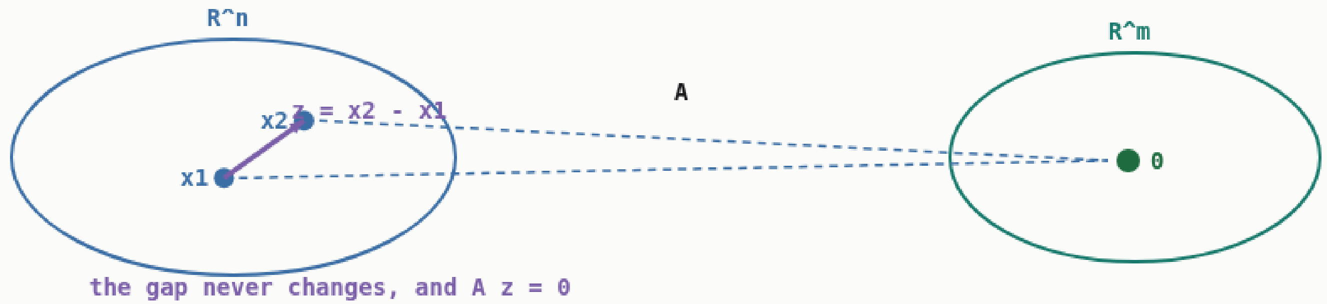
$$\begin{aligned} Ax_1 &= Ax_2 \\ Ax_1 - Ax_2 &= 0 \\ A(x_1 - x_2) &= 0 \end{aligned}$$

`last step` linearity, and nothing else, is what pulls A outside

**THE DIFFERENCE** $\boxed{z = x_1 - x_2 \implies Az = 0}$

`z` nonzero, since $x_1 \neq x_2$, and A sends it to zero

> Every collision $Ax_1 = Ax_2$ is a solution $z = x_1 - x_2$ of $Ax = 0$.




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.1 linear map, p. 52; 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

HOMOGENEOUS SYSTEM $\boxed{Ax = 0}$

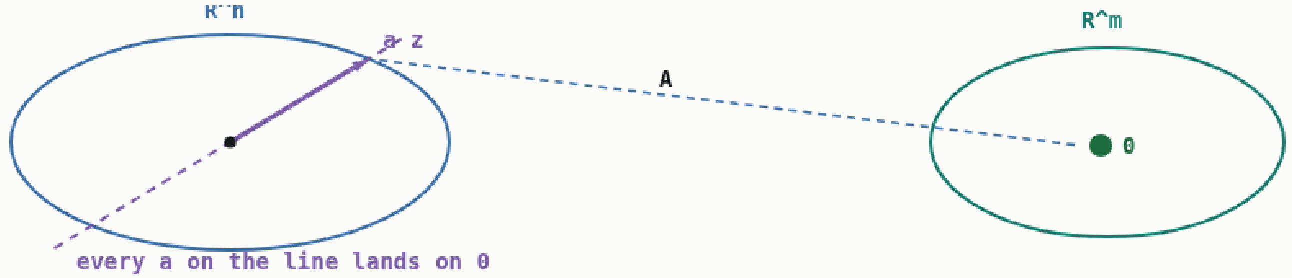
`homogeneous` the right-hand side is the zero vector

`x = 0` always a solution, and the uninteresting one

> Are there nonzero solutions?

ONE GIVES A LINE $\boxed{Az = 0,\ z \neq 0 \implies A(\alpha z) = 0\ \forall\, \alpha \in \mathbb{R}}$

`αz` one invisible direction, so a whole line of invisible inputs




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.26 homogeneous system of linear equations, p. 65. Springer, 2024.

# Null Space and Linear Independence

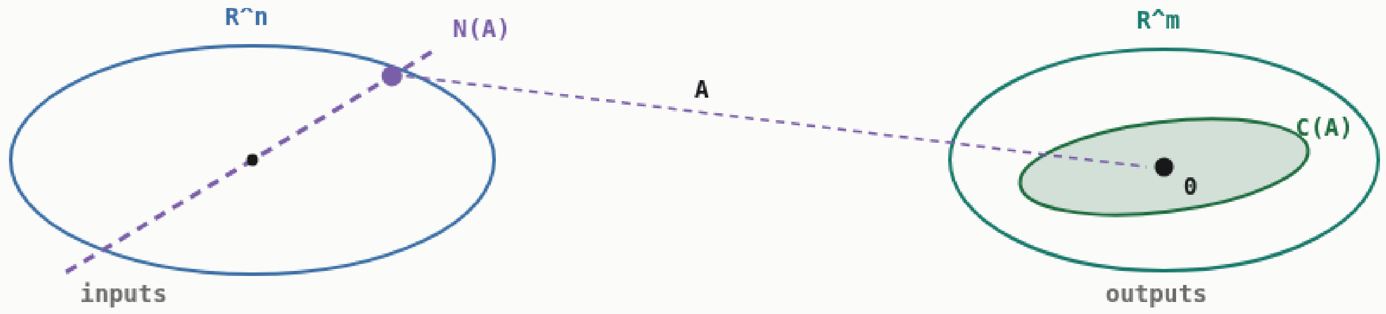
NULL SPACE $\boxed{\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}}$

also called the kernel of A, written $\ker(A)$

$Ax = 0$ the inputs A cannot see

SHAPE $\boxed{\mathcal{N}(A) \subseteq \mathbb{R}^n}$ against $\mathcal{C}(A) \subseteq \mathbb{R}^m$

input side N(A) lives where x lives, C(A) where Ax lives




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space and 3.13 the null space is a subspace, p. 59. Springer, 2024.

# Null Space and Linear Independence

$$u, v \in \mathcal{N}(A) \implies Au = 0,\ Av = 0$$

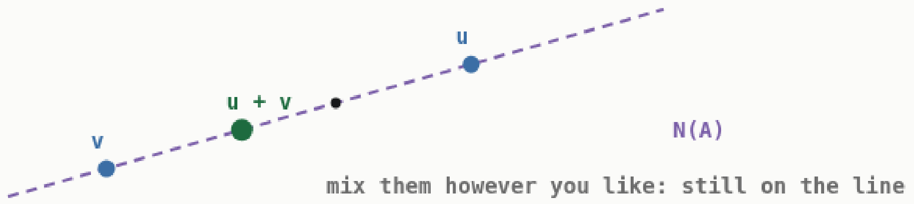
and $A0 = 0$, so the origin is in $\mathcal{N}(A)$ to begin with

$$A(\alpha u + \beta v) = \alpha Au + \beta Av = 0$$

so $\alpha u + \beta v \in \mathcal{N}(A)$ for every $\alpha, \beta$

SUBSPACE $\boxed{\mathcal{N}(A) \text{ is a subspace of } \mathbb{R}^n}$

flat in this $\mathbb{R}^3$ example: a point, a line, or a plane through 0




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.13 the null space is a subspace, p. 59. Springer, 2024.

# Null Space and Linear Independence

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

OUR MATRIX $a_3 = a_1 + a_2 \implies a_1 + a_2 - a_3 = 0$

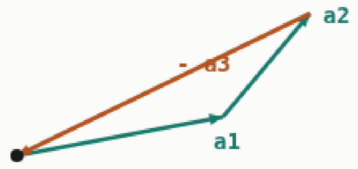
read it a mixture of the columns, weights $1, 1, -1$, giving zero

$$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0 \implies \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in \mathcal{N}(A)$$

and the next page solves for all of it

THE WHOLE SET $\boxed{\mathcal{N}(A) = \operatorname{span}\left\{(1, 1, -1)^{\mathsf{T}}\right\}}$

a line through the origin




a1 + a2 - a3 walks back to 0

so (1, 1, -1) is in N(A)

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

Ax = 0 $$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = 0$$

row 1 $1x_1 + 0x_2 + 1x_3 = 0$, so $x_1 + x_3 = 0$

row 2 $0x_1 + 1x_2 + 1x_3 = 0$

row 3 $2x_1 + x_2 + 3x_3 = 0$ is twice row 1 plus row 2

SOLVE $$x_3 = t:\ x_1 = -t,\ x_2 = -t,\ x = t(-1, -1, 1)^\mathsf{T}$$

why one free unknown

NULL SPACE $$\boxed{\mathcal{N}(A) = \operatorname{span}\{(-1, -1, 1)^\mathsf{T}\}}$$

check $-1 + 1,\ -1 + 1,\ -2 - 1 + 3$: all zero

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

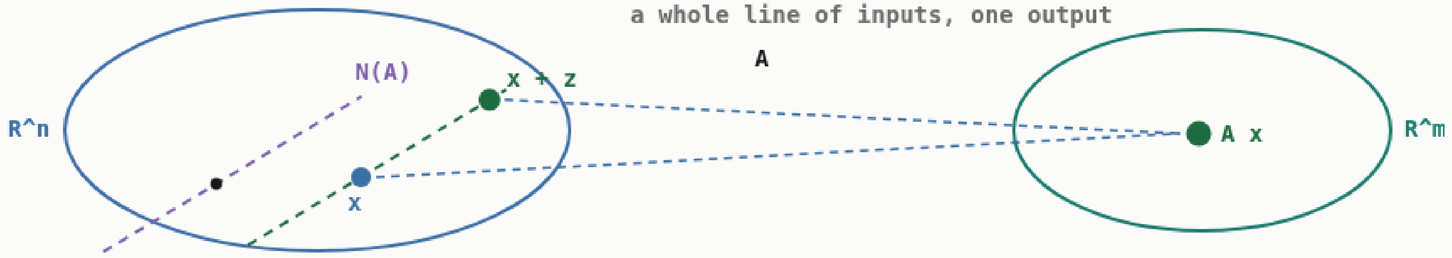
$$z \in \mathcal{N}(A) \implies Az = 0 \implies A(x+z) = Ax + Az = Ax$$

`x` any input at all, not a special one

WHAT IT MEANS $\boxed{A(x+z) = Ax, \qquad z \in \mathcal{N}(A)}$

`x + z` a whole line of inputs, all landing on one output

> Null-space directions are invisible to the map.




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.1 linear map, p. 52; 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

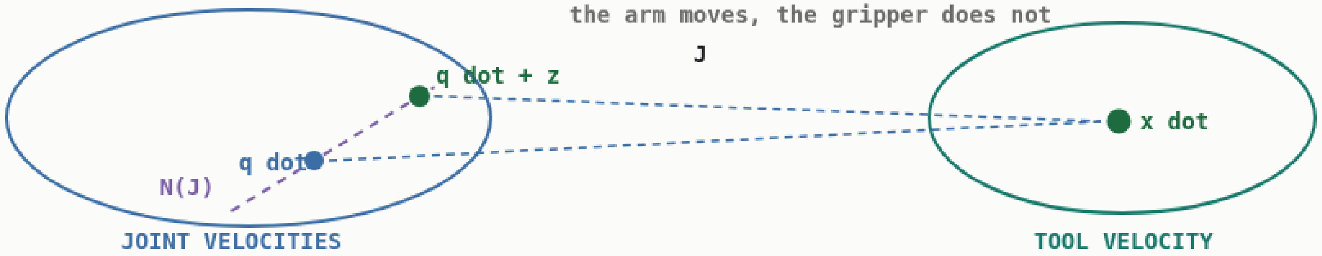
ON A ROBOT $\dot{x} = J(q)\,\dot{q}$

$\dot{q}$ joint velocities, what you command

$\dot{x}$ end-effector velocity, what the world sees

REDUNDANCY $\boxed{z \in \mathcal{N}(J(q)) \implies J(q)(\dot{q} + z) = J(q)\dot{q}}$

z a joint-velocity direction that produces no end-effector velocity




Lynch, K. M. and Park, F. C. ***Modern Robotics***, §6.3 inverse velocity kinematics, p. 233. Cambridge University Press, 2017 preprint.

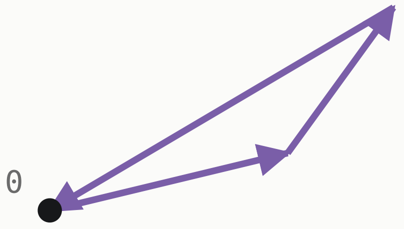
# Null Space and Linear Independence

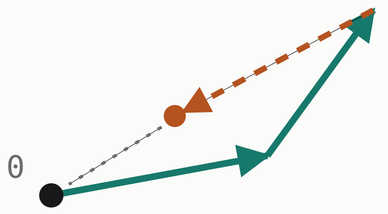
LINEAR INDEPENDENCE $\boxed{c_1 v_1 + \cdots + c_k v_k = 0 \implies c_1 = \cdots = c_k = 0}$

read it the all-zero weights are the only ones that cancel

SAME THING $\boxed{\sum_{i=1}^{k} c_i v_i = 0 \implies c = 0}$

c the whole list of weights, read as one vector




Left, a nonzero walk comes home. Right, none does.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.15 linearly independent, p. 32. Springer, 2024.

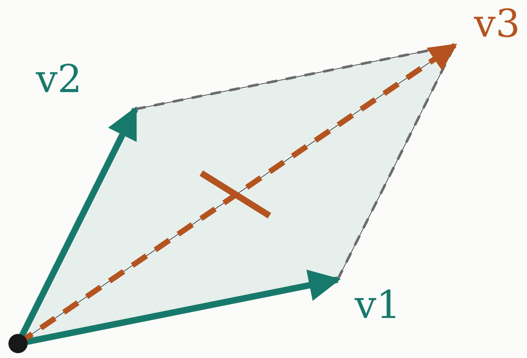
# Null Space and Linear Independence

LINEAR DEPENDENCE $\boxed{\exists\, c \neq 0 : \; c_1 v_1 + \cdots + c_k v_k = 0}$

$c \neq 0$ at least one weight is nonzero; say it is $c_j$

SOLVE FOR THAT ONE $\boxed{v_j = -\frac{1}{c_j} \sum_{i \neq j} c_i v_i}$

SO $v_j$ is built from the others, and deleting it leaves the span unchanged




DELETE v3

SAME PLANE

Strike the diagonal and the plane is untouched.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.17 linearly dependent and 2.19 linear dependence lemma, p. 33. Springer, 2024.

# Null Space and Linear Independence

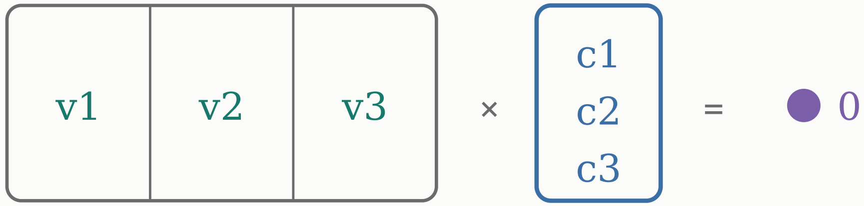
STACK THEM $V = [\, v_1 \ \cdots \ v_k \,], \qquad Vc = \sum_i c_i v_i$

`Vc` the mixture, written as a matrix times its weights

THE ANCHOR

$$\boxed{\begin{aligned} v_1, \ldots, v_k \text{ independent} &\iff \mathcal{N}(V) = \{0\} \\ v_1, \ldots, v_k \text{ dependent} &\iff \mathcal{N}(V) \neq \{0\} \end{aligned}}$$

`one question` redundancy in a list is a null space




c lives in N(V)

INDEPENDENT MEANS
c = 0 IS THE ONLY ONE

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.15 linearly independent, p. 32; 3.15 injectivity, p. 60. Springer, 2024.

# Null Space and Linear Independence

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

OUR MATRIX $a_3 = a_1 + a_2 \implies A(1, 1, -1)^\mathsf{T} = 0$

one equation the weights $1, 1, -1$ cancel the three columns

NULL SPACE

$$(1, 1, -1)^\mathsf{T} \in \mathcal{N}(A)$$

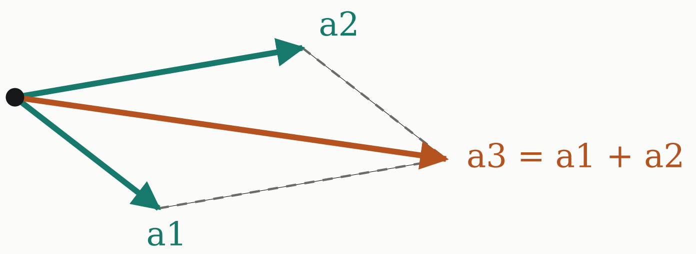
reading an input A cannot see

LINEAR DEPENDENCE

$$a_1, a_2, a_3 \text{ dependent}$$

reading a column you may delete

> Same equation, two interpretations.




THE THIRD COLUMN
BUYS NOTHING

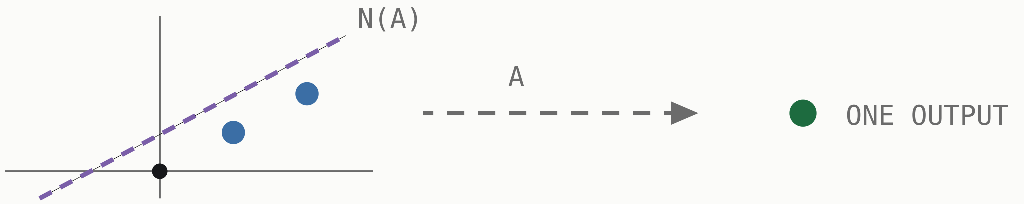
Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.17 linearly dependent, p. 33; 3.11 null space, p. 59. Springer, 2024.

# Null Space and Linear Independence

$$\boxed{\begin{aligned} \mathcal{N}(A) &= \{x : Ax = 0\} \\ z \in \mathcal{N}(A) &\implies A(x+z) = Ax \\ \mathcal{N}(A) \neq \{0\} &\implies \text{input information is lost} \end{aligned}}$$

the chain definition, meaning, cost

FOR THE COLUMNS $\boxed{\mathcal{N}(A) = \{0\} \iff \text{columns independent}}$

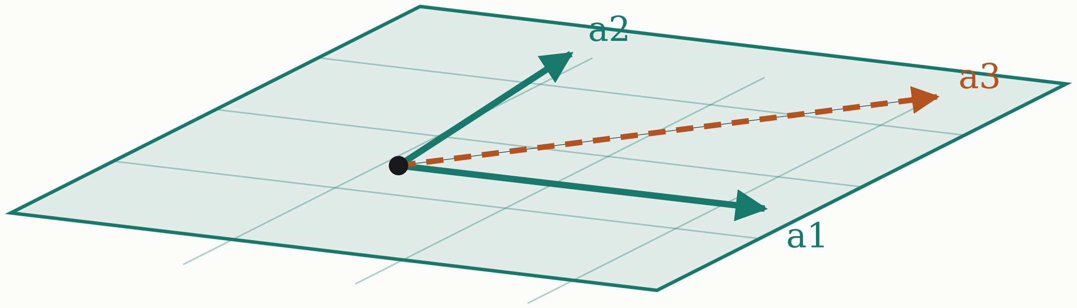



settled whether any direction is redundant

> How many independent directions remain?

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.26 basis, p. 39. Springer, 2024.

# Basis, Dimension, and Rank




HOW MANY
SURVIVE?

# Basis, Dimension, and Rank

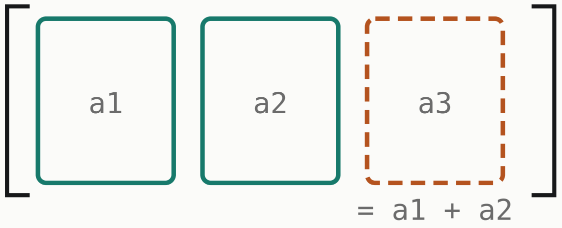
$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

WE KNOW $\mathcal{C}(A)$ what A can produce, $\mathcal{N}(A)$ what A cannot see

and how to tell whether a direction is redundant

NOW ASK **How many independent directions are needed?**

our A $a_3 = a_1 + a_2$: three columns, but how many directions?



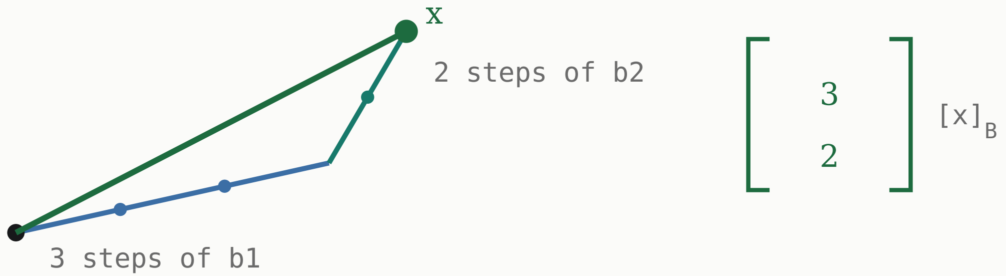

THREE COLUMNS

HOW MANY DIRECTIONS?

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.26 basis, p. 39. Springer, 2024.

# Basis, Dimension, and Rank

BASIS $\boxed{\text{Basis} = \text{Spanning} + \text{Linear Independence}}$




**spans** $\mathrm{span}(b_1, \ldots, b_k) = V$

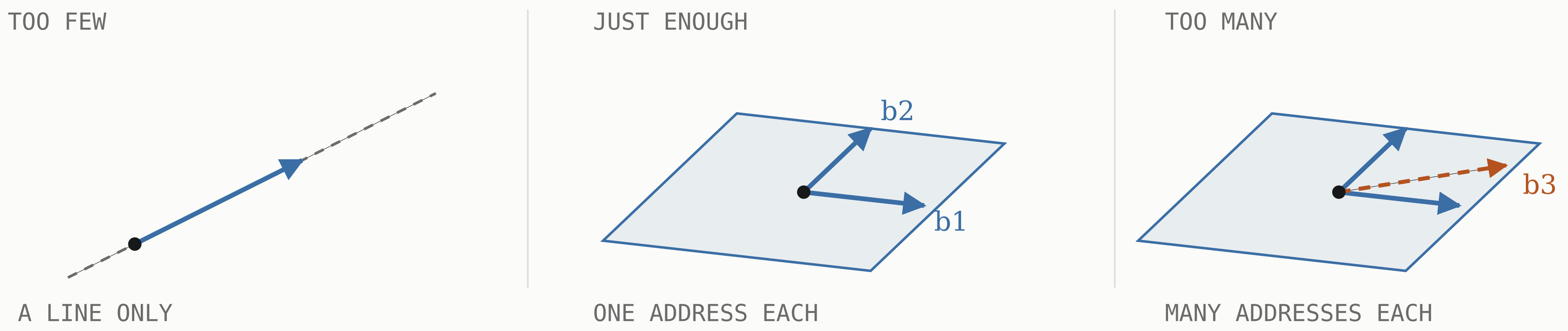
**independent** $c_1 b_1 + \cdots + c_k b_k = 0 \Rightarrow c_1 = \cdots = c_k = 0$

SO enough to build everything, with nothing spare

**both halves** either one alone is not a basis

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.26 basis, p. 39. Springer, 2024.

# Basis, Dimension, and Rank




Too few cannot reach; too many give several addresses.

**BASIS** $\boxed{\text{enough, but not too many}}$

`too few` here independent, but does not reach everything

`too many` reaches everything, but carries a repeat

**WORKED EXAMPLE**

In $\mathbb{R}^2$: $\{(1,0)\}$ is too few, $\{(1,0),(1,1)\}$ is a basis, adding $(2,1)$ is too many.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.26 basis, p. 39. Springer, 2024.

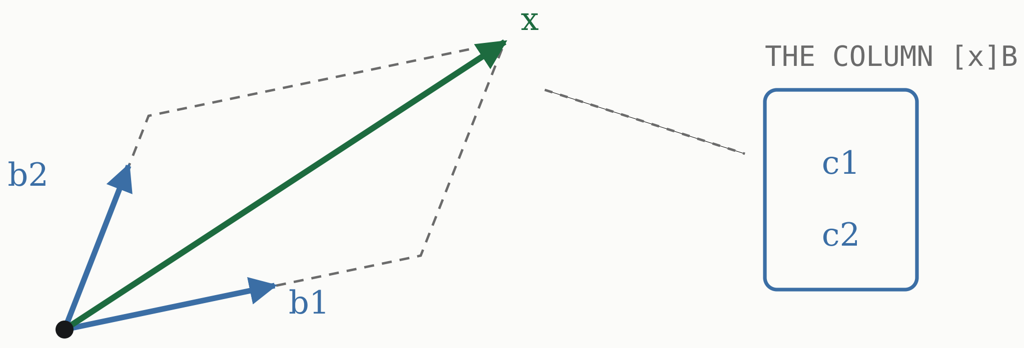
# Basis, Dimension, and Rank

**UNIQUE WEIGHTS** $x = c_1 b_1 + \cdots + c_k b_k$ in exactly one way

why independence: two different weightings would differ by a nonzero mix that vanishes

**COORDINATES** $\boxed{[x]_B = (c_1, \ldots, c_k)^{\mathsf{T}}, \qquad x = B\,[x]_B}$

one address the vector does not change; only its coordinates do




Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.28 criterion for basis, p. 39; 3.73 matrix of a vector, p. 88. Springer, 2024.

# Basis, Dimension, and Rank

DIMENSION $\boxed{\dim V = \text{the number of vectors in any basis of } V}$

| a line | a plane | all of $\mathbb{R}^3$ |
| --- | --- | --- |

one more independent direction each time

any every basis of the same space has the same length

READING dimension counts independent degrees of freedom

finite every space in this course has a finite basis

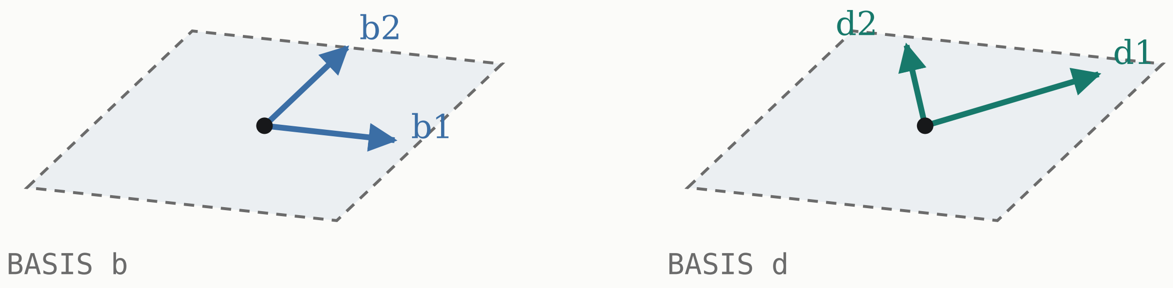
$$\boxed{\dim\{0\} = 0}$$

zero space the empty list is a basis of it

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.35 dimension, p. 44. Springer, 2024.

# Basis, Dimension, and Rank

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$




One plane, two bases, both of size two.

SAME SIZE ALWAYS $\boxed{\text{different bases, different arrows, the same number of them}}$

this plane every basis of it has exactly two vectors

so dimension belongs to the space, not to a choice

Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.35 dimension, p. 44. Springer, 2024.

# Basis, Dimension, and Rank

REDUCE $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - 2r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 \leftarrow r_3 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$

step 1 $(2,1,3) - 2(1,0,1) = (0,1,1)$

step 2 $(0,1,1) - (0,1,1) = (0,0,0)$

pivots columns 1 and 2

WHY R $R = EA,\ E \text{ invertible}:\ Ax = 0 \iff Rx = 0$

so $\mathcal{N}(A) = \mathcal{N}(R)$

Strang, G. ***18.06***, Lecture 7: pivot variables and special solutions. MIT OCW, 2010.

# Basis, Dimension, and Rank

**PIVOT** $\boxed{\text{the leading nonzero entry of a nonzero row of } R}$

**in R** pivot positions are read off the echelon form, not off $A$

**pivot variable** a variable whose column carries a pivot

**free variable** a variable whose column does not

$$R = \begin{array}{c} \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x1 & x2 & x3 \end{array} \end{array}$$

pivots

free

**HERE** $x_1, x_2$ pivot; $x_3$ free

**count** 3 variables, 2 pivots, 1 free

Strang, G. ***18.06***, Lecture 7: pivot variables and special solutions. MIT OCW, 2010.

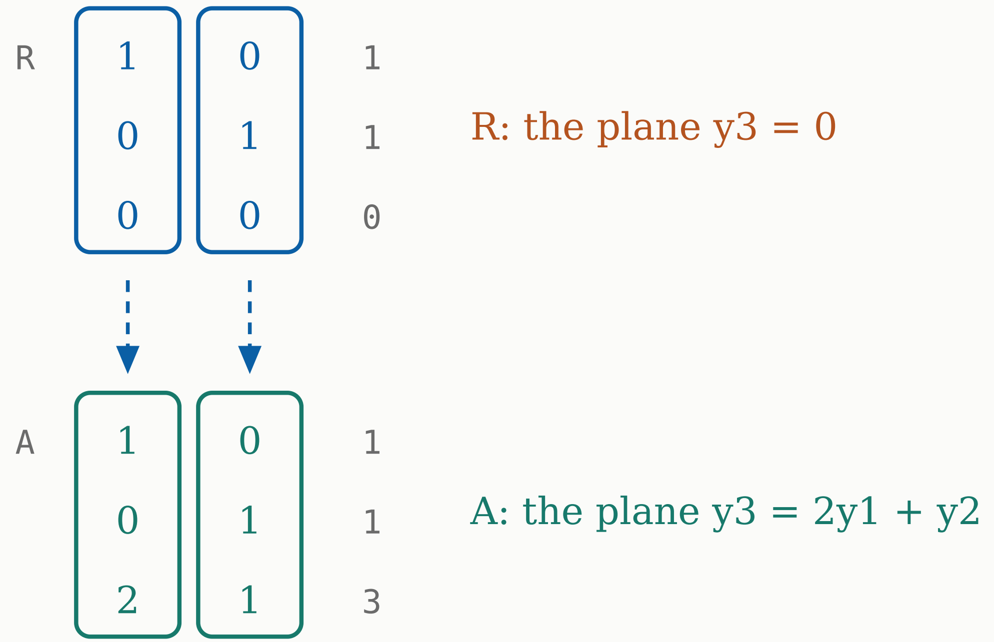
# Basis, Dimension, and Rank

PIVOT COLUMNS $\boxed{\text{the pivot columns of } A \text{ are a basis for } \mathcal{C}(A)}$

but $\mathcal{N}(A) = \mathcal{N}(R)$, while $\mathcal{C}(A) \neq \mathcal{C}(R)$

R's columns $(1,0,0)^\top, (0,1,0)^\top$ span $y_3 = 0$; $a_1 = (1,0,2)^\top$ has $y_3 = 2$

A's columns $a_1, a_2$ independent, and $a_3 = a_1 + a_2$, so they span $\mathcal{C}(A)$




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.52 column rank, p. 77. Springer, 2024.

# Basis, Dimension, and Rank



ROWS OF R $x_1 + x_3 = 0, \qquad x_2 + x_3 = 0, \qquad 0 = 0$

legal $\mathcal{N}(A) = \mathcal{N}(R)$

free column 3 has no pivot: $x_3 = t$

SOLVE $x_1 = -t, \quad x_2 = -t \implies x = (-t, -t, t)^\mathsf{T} = t(-1, -1, 1)^\mathsf{T}$

$t = 1$ on $A$: $-1 + 1,\ -1 + 1,\ -2 - 1 + 3$, all zero

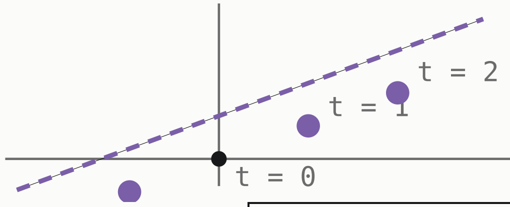

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Basis, Dimension, and Rank

ALL SOLUTIONS $\mathcal{N}(A) = \{\, t(-1,-1,1)^{\mathsf{T}} :\ t \in \mathbb{R} \,\}$

why every solution is $t$ times one vector, nothing else

so all multiples of one vector: a span




BASIS AND DIMENSION $\boxed{\mathcal{N}(A) = \operatorname{span}\{(-1,-1,1)^{\mathsf{T}}\}, \qquad \dim \mathcal{N}(A) = 1}$

basis one nonzero vector is independent on its own, so it is a basis

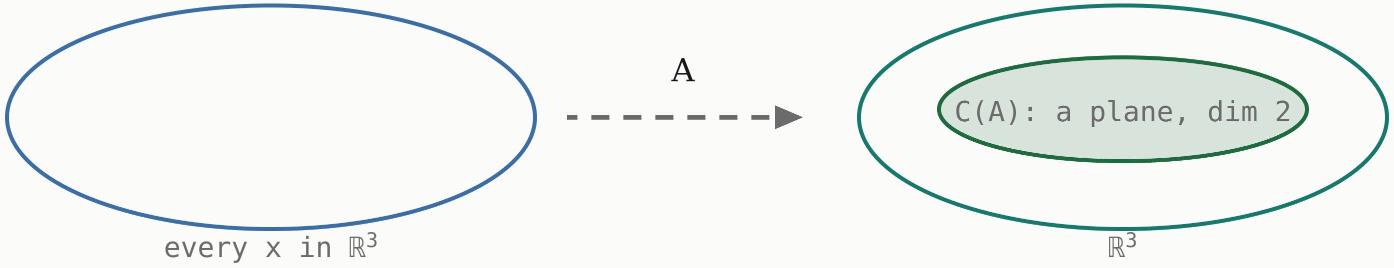
count one free variable, one basis vector: $\dim \mathcal{N}(A) =$ free variables

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59; 2.26 basis, p. 39. Springer, 2024.

# Basis, Dimension, and Rank

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

**RANK** $\boxed{\mathrm{rank}(A) = \dim \mathcal{C}(A)}$




our A $\dim \mathcal{C}(A) = 2$, so $\mathrm{rank}(A) = 2$

**TO COMPUTE** $\boxed{\mathrm{rank}(A) = \text{the number of pivots}}$

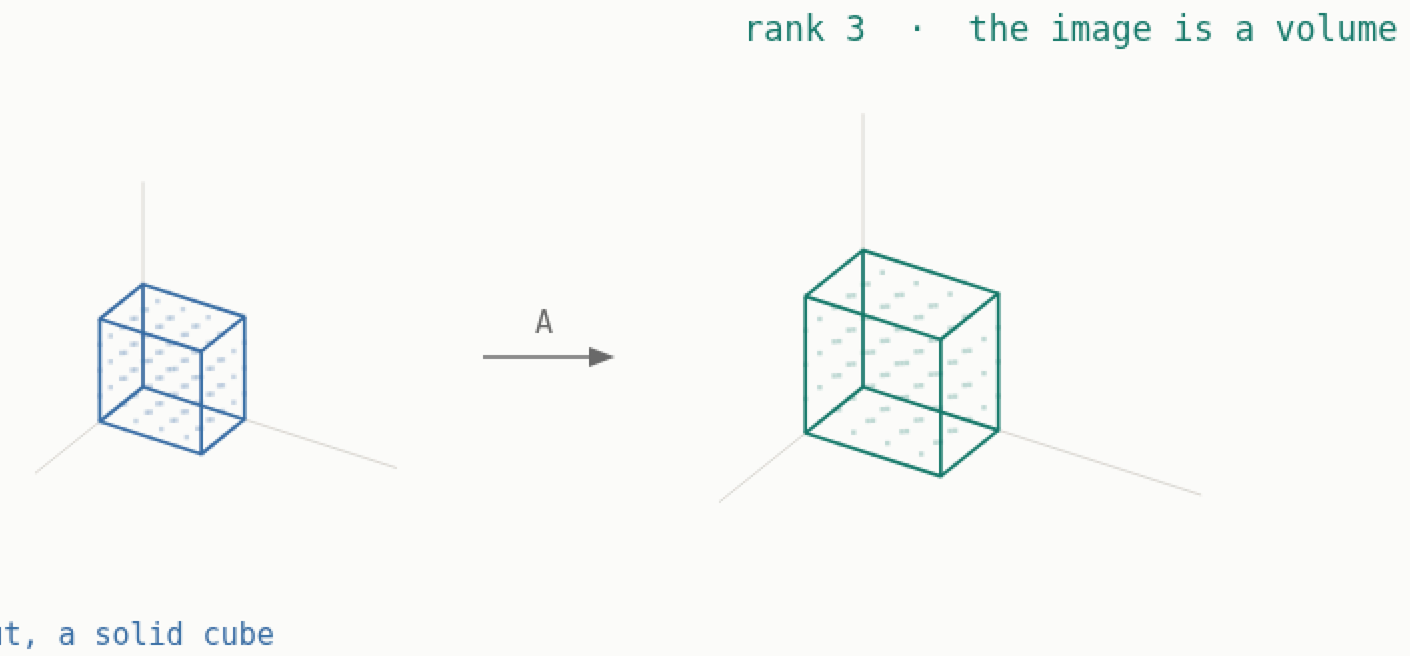
reading how many independent output directions A can produce

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.58 rank, p. 79; 3.52 column rank, p. 77. Springer, 2024.

# Basis, Dimension, and Rank

INTO THREE-SPACE $r = 3$ volume; $r = 2$ plane; $r = 1$ line; $r = 0$ the point $\{0\}$

rank the dimension of the reachable output space



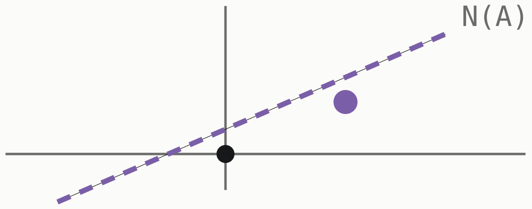

The input cube never changes. Each direction the matrix loses costs the image a whole dimension.

Sanderson, G. ***Essence of Linear Algebra***, ch. 7. 3Blue1Brown, 2016.

# Basis, Dimension, and Rank

A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

**NULLITY** $\boxed{\text{nullity}(A) = \dim \mathcal{N}(A)}$




ONE FREE DIRECTION

nullity(A) = 1

our A $\mathcal{N}(A) = \text{span}\{(-1,-1,1)^\top\}$, so $\text{nullity}(A) = 1$

**READING** independent input directions lost by $A$

the pair rank counts what survives, nullity counts what disappears

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Basis, Dimension, and Rank

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

RANK–NULLITY $\boxed{\operatorname{rank}(A) + \operatorname{nullity}(A) = n}$

| | |
|---|---|
| n | the number of columns, the input dimension |
| reach | $\dim \mathcal{C}(A) = r$: directions that survive |
| loss | $\dim \mathcal{N}(A) = n - r$: directions that disappear |
| our A | $2 + 1 = 3$ |
| a choice | $n - r$ basis vectors in $\mathcal{N}(A)$, $r$ whose images span $\mathcal{C}(A)$ |

| r = 2 | n - r = 1 |
|---|---|
| 2 survive | 1 dies |

n = 3 columns of A

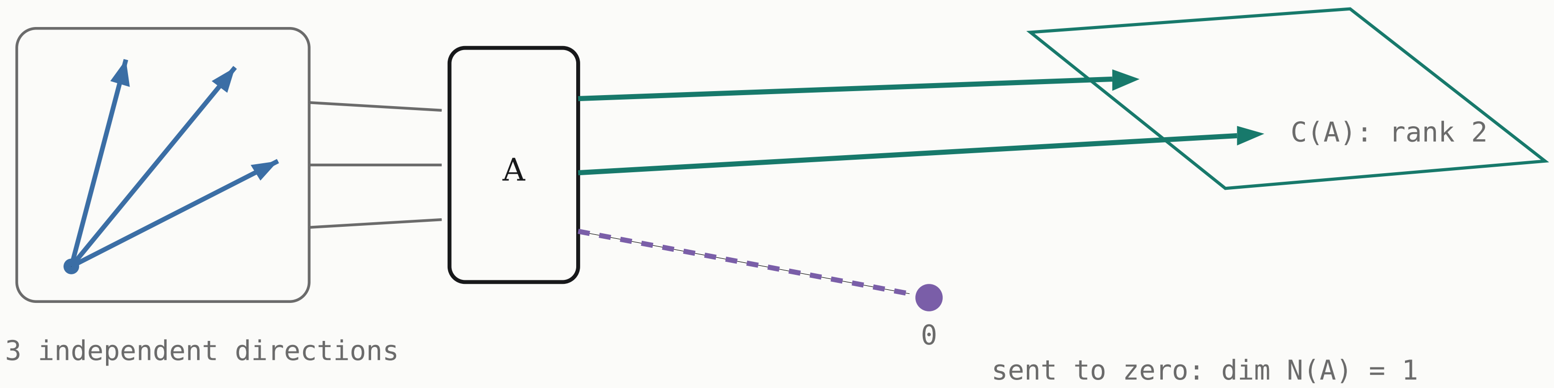
Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Basis, Dimension, and Rank

**THE MAP** $A : \mathbb{R}^n \to \mathbb{R}^m, \qquad n = r + (n - r)$

`r` directions that survive

`n - r` directions that vanish




Two survivors, one casualty, three directions accounted for.

**SO** $\boxed{\text{rank–nullity counts the input side}}$

`not m` the null space is not in the output space

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Basis, Dimension, and Rank

EXAM QUESTION $B \in \mathbb{R}^{7\times 10}, \qquad \operatorname{rank}(B) = 6$

ask $\dim \mathcal{C}(B)$, $\dim \mathcal{N}(B)$, and the number of free variables

$$\dim \mathcal{C}(B) = 6, \qquad \dim \mathcal{N}(B) = 10 - 6 = 4, \qquad 4 \text{ free variables}$$

the trap use the 10, not the 7

B

7 rows × 10 columns

THE TEN INPUT DIRECTIONS

6 SURVIVE = rank

4 GO TO ZERO

? → 10 − 6

USE THE 10,
NOT THE 7

$$\dim \mathcal{N}(B) = n - r = 10 - 6 = 4.$$

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Basis, Dimension, and Rank

FULL COLUMN RANK

$$\boxed{r = n}$$

then $\mathcal{N}(A) = \{0\}$: no input direction is lost; needs $n \le m$

all n inputs survive

N(A) = {0}

FULL ROW RANK

$$\boxed{r = m}$$

then $\mathcal{C}(A) = \mathbb{R}^m$: every output is reachable; needs $m \le n$

all m outputs reached

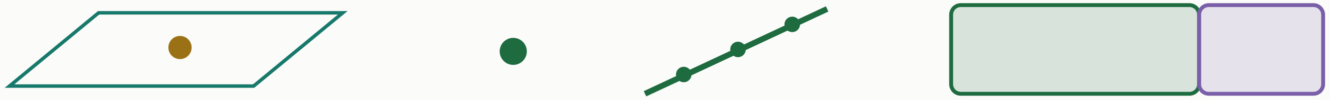
C(A) = $\mathbb{R}^m$

> One is about losing nothing. The other is about reaching everything.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Basis, Dimension, and Rank

WHERE WE ARE $\dim \mathcal{C}(A) = r, \qquad \dim \mathcal{N}(A) = n - r$



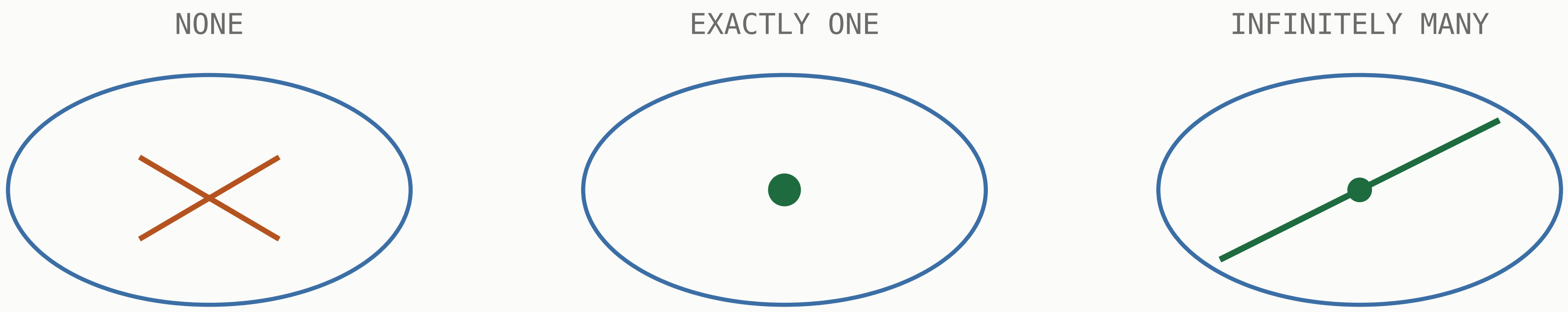
so the three questions become countable

| | |
|---|---|
| is $b$ reachable? | $b \in \mathcal{C}(A)$ |
| is $x$ unique? | $\mathcal{N}(A) = \{0\}$ |
| how many free directions? | $n - r$ |

row two uniqueness is a question only once a solution exists

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Linear Systems and Solution Sets



# Linear Systems and Solution Sets

RETURN TO $\boxed{Ax = b}$


1 existence does a solution exist?

2 uniqueness if it exists, is it the only one?

3 structure what are all of them?

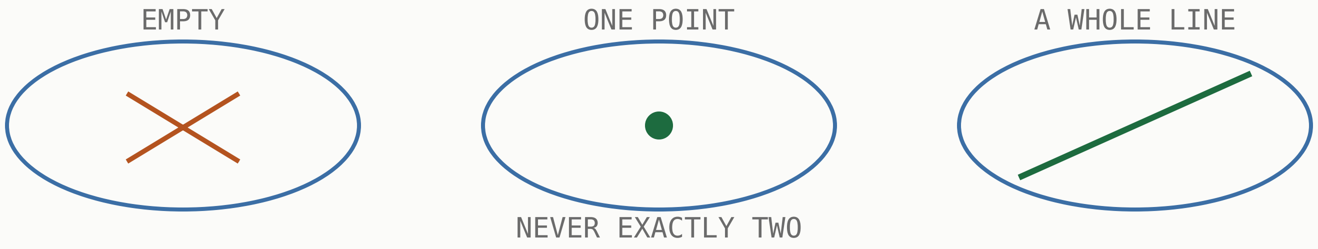
WITH $\mathcal{C}(A)$ and $\mathcal{N}(A)$

nothing new both spaces are already built and measured

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range, p. 61; 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

**SOLUTION SET** $\boxed{\mathcal{S}(A,b) = \{x \in \mathbb{R}^n : Ax = b\}}$




`a solution` any x that produces the requested b

`consistent` $\mathcal{S}(A,b) \neq \emptyset$; otherwise inconsistent

**THREE POSSIBILITIES** empty, one point, infinitely many

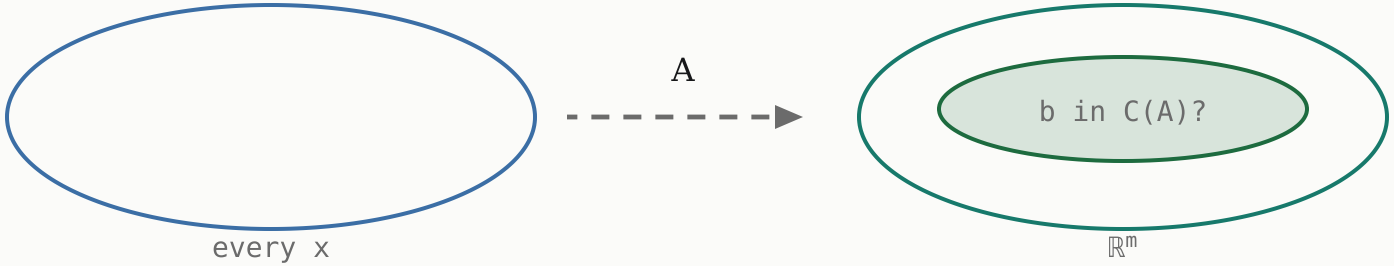
`never two` two solutions force infinitely many: a line, a plane, or higher

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

RECALL $\mathcal{C}(A) = \{Ax : x \in \mathbb{R}^n\}$




so b is reachable exactly when some x reaches it

EXISTENCE $\boxed{Ax = b \text{ is consistent } \iff b \in \mathcal{C}(A)}$

one sentence the column space controls existence

**WORKED EXAMPLE**

$\mathcal{C}(A)$ is $y_3 = 2y_1 + y_2$: $(3, 2, 8)$ passes since $2(3) + 2 = 8$; $(3, 2, 9)$ fails.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.16 range, p. 61. Springer, 2024.

# Linear Systems and Solution Sets

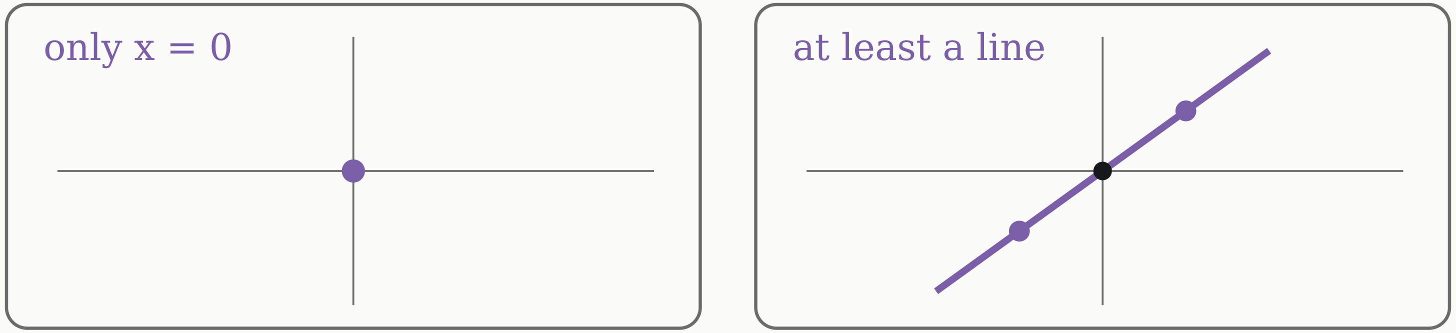
SUPPOSE $Ax_1 = b, \quad Ax_2 = b \implies A(x_1 - x_2) = 0 \implies x_1 - x_2 \in \mathcal{N}(A)$

so if $\mathcal{N}(A) = \{0\}$ then $x_1 = x_2$

UNIQUENESS $\boxed{\mathcal{N}(A) = \{0\} \implies \text{at most one solution}}$

careful at most one, not one

if consistent then it means exactly one

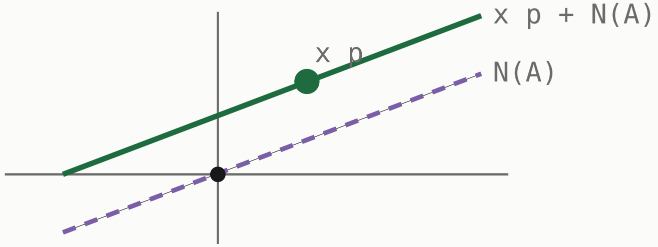



Either only the origin, or at least a whole line of erased inputs.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.15 injectivity, p. 60. Springer, 2024.

# Linear Systems and Solution Sets

**PARTICULAR** $\boxed{Ax_p = b \text{ for one particular } x_p}$




ONE SOLUTION, THEN
EVERYTHING A IGNORES

`any one` it does not matter which solution you found

**GENERAL** $\boxed{\mathcal{S}(A, b) = x_p + \mathcal{N}(A) = \{x_p + z : z \in \mathcal{N}(A)\}}$

`recipe` find one solution, then add everything A ignores

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

**EACH ONE WORKS**

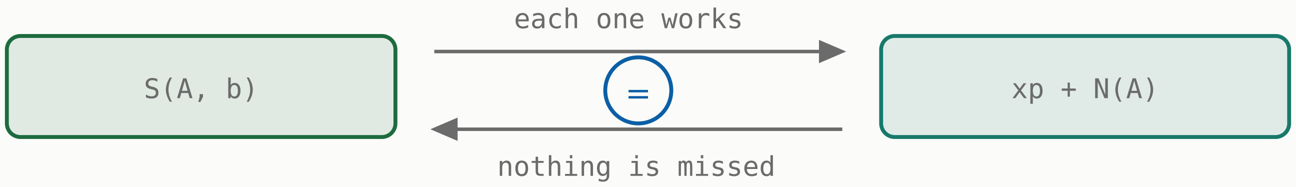
$$A(x_p + z) = b + 0 = b$$

uses linearity and $Az = 0$

**NOTHING IS MISSED**

$$Ax = b \Rightarrow A(x - x_p) = 0$$

so $x = x_p + z$ with $z = x - x_p \in \mathcal{N}(A)$




**EQUALITY** $\boxed{\mathcal{S}(A, b) = x_p + \mathcal{N}(A)}$

both ways one inclusion each, which is what an equality of sets needs

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

AUGMENT

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 3 & 8 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

step 1 $(2, 1, 3|8) - 2(1, 0, 1|3) = (0, 1, 1|2)$

step 2 $(0, 1, 1|2) - (0, 1, 1|2) = (0, 0, 0|0)$

row 3 $0 = 0$: consistent

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

READ OFF $x_1 + x_3 = 3, \quad x_2 + x_3 = 2, \quad x_3 = t \implies x_1 = 3 - t,\ x_2 = 2 - t$

free no pivot in column 3: choose $x_3$

$t = 0$ one particular solution, $x_p = (3, 2, 0)^\mathsf{T}$

ALL OF THEM $\boxed{x = (3, 2, 0)^\mathsf{T} + t\,(-1, -1, 1)^\mathsf{T}, \qquad t \in \mathbb{R}}$

$t = 1$ $(2, 1, 1)^\mathsf{T}$: $2 + 1 = 3$, $1 + 1 = 2$, $4 + 1 + 3 = 8$

$b = (3, 2, 9)^\mathsf{T}$ same steps: $9 - 6 = 3$, $3 - 2 = 1$, so row 3 reads $0 = 1$: no solution

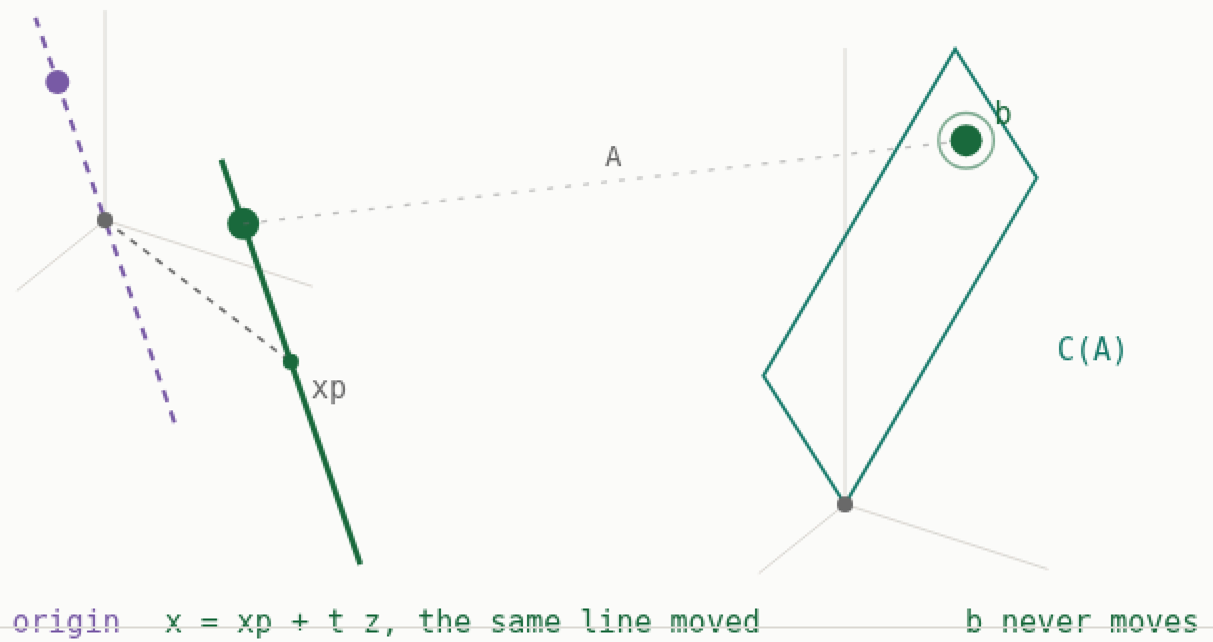
Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.11 null space, p. 59. Springer, 2024.

# Linear Systems and Solution Sets

AFFINE SET $\boxed{x_0 + S = \{x_0 + s : s \in S\} \text{ for a subspace } S}$

here $\mathcal{S}(A, b) = x_p + \mathcal{N}(A)$: the null space, translated

through 0? only if $b = 0$; then $\mathcal{S} = \mathcal{N}(A)$, a subspace




Axler, S. ***Linear Algebra Done Right***, 4th ed., 1.34 conditions for a subspace, p. 18. Springer, 2024.

The null space through the origin, the same line moved to a particular solution, and an output that never moves.

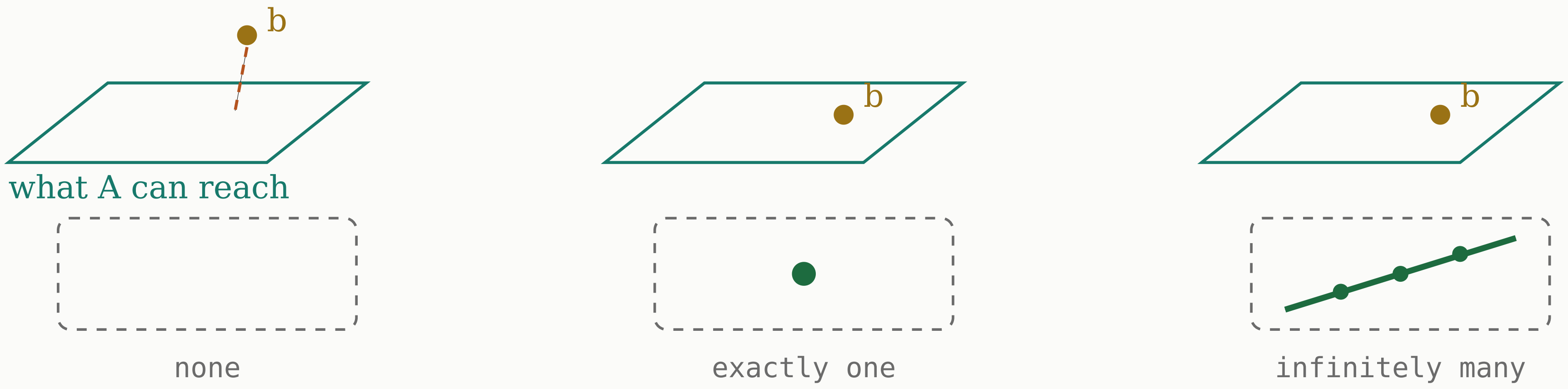
# Linear Systems and Solution Sets

NOTATION $r = \operatorname{rank}(A)$

unique when consistent, means $r = n$

CASES
$$\boxed{\begin{aligned} b \notin \mathcal{C}(A):&\ \text{none} \\ b \in \mathcal{C}(A),\ \mathcal{N}(A) = \{0\}:&\ \text{one} \\ b \in \mathcal{C}(A),\ \dim \mathcal{N}(A) > 0:&\ \text{infinitely many} \end{aligned}}$$

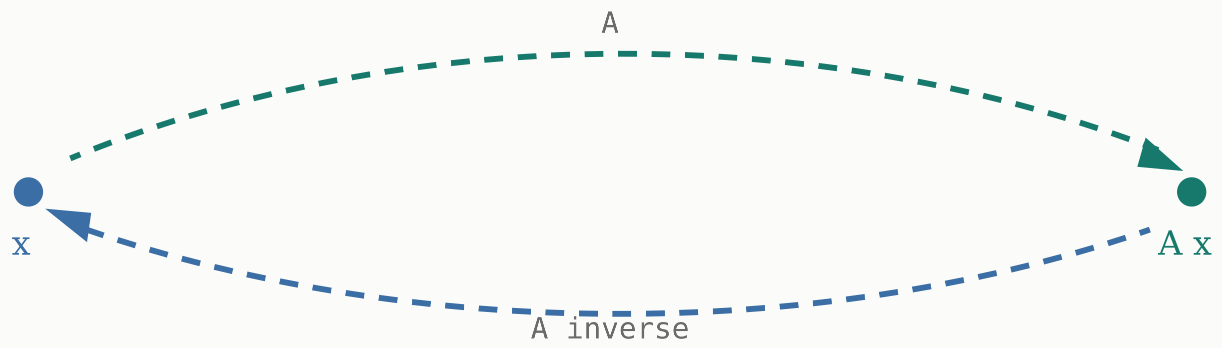
free $n - r$ directions




Top: the demand. Bottom: the solutions.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Invertibility and Singular Matrices



# Invertibility and Singular Matrices

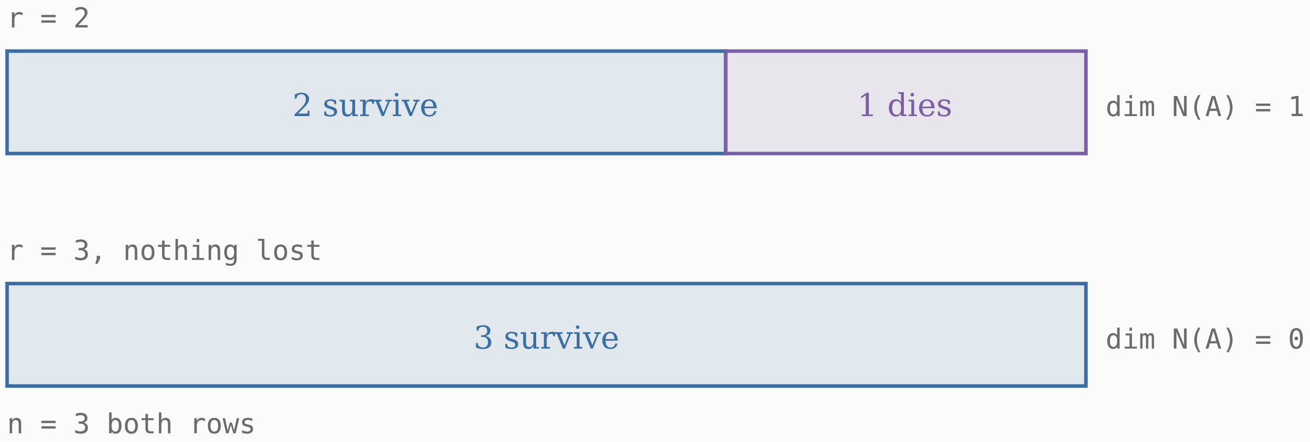
**DEMAND** $\mathcal{C}(A) = \mathbb{R}^m$ and $\mathcal{N}(A) = \{0\}$

`reach` $r = m$: nothing is missed

`no loss` $r = n$ by rank--nullity

**SO** $\boxed{m = n \text{ and } r = n}$

`square` a conclusion, not a hypothesis




Both cases have three input directions to spend. Only the top spends one on the null space.


Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21 fundamental theorem of linear maps, p. 62. Springer, 2024.

# Invertibility and Singular Matrices

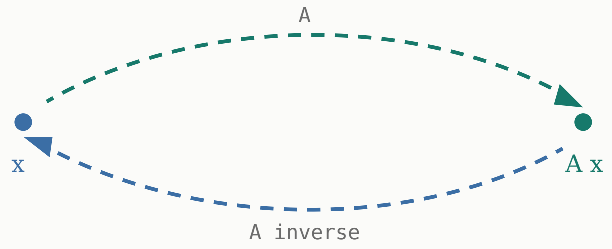
**INVERSE** $\boxed{AA^{-1} = A^{-1}A = I}$

**shapes** $A, A^{-1} \in \mathbb{R}^{n\times n}$ and $I = I_n$

**both sides** two-sided by definition; for square $A$ either half implies the other

**SINGULAR** $\boxed{A \text{ singular} \iff A \text{ has no inverse}}$

**nonsingular** the same word




**WORKED EXAMPLE**

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.79 identity, p. 90; 3.80 invertible and the word singular, p. 91; 3.60 inverse is unique, p. 82. Springer, 2024.

# Invertibility and Singular Matrices

IN GENERAL $A$ invertible $\iff$ $A$ injective and surjective

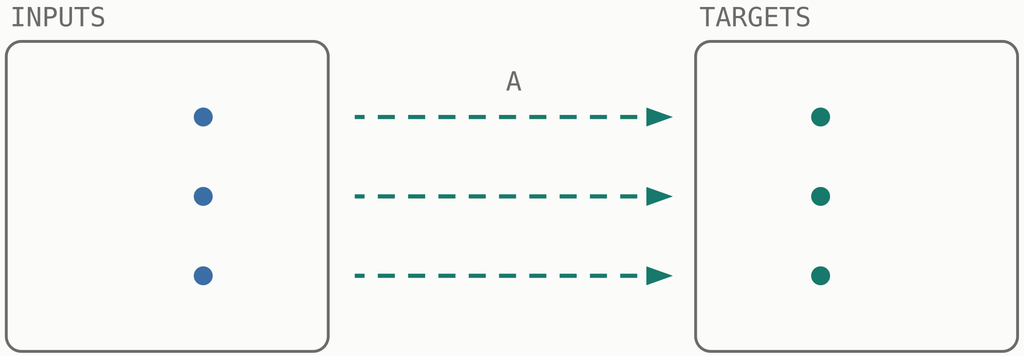
injective $Ax = Ay \Rightarrow x = y$

surjective $\mathcal{C}(A) = \mathbb{R}^m$

IF A IS SQUARE $\boxed{\mathcal{N}(A) = \{0\} \iff \mathcal{C}(A) = \mathbb{R}^n \iff A \text{ invertible}}$

saving check one side, the other follows

why rank--nullity: $r = n$ makes both true at once




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.63 invertibility, p. 83; 3.65 injectivity equivalent to surjectivity, p. 84. Springer, 2024.

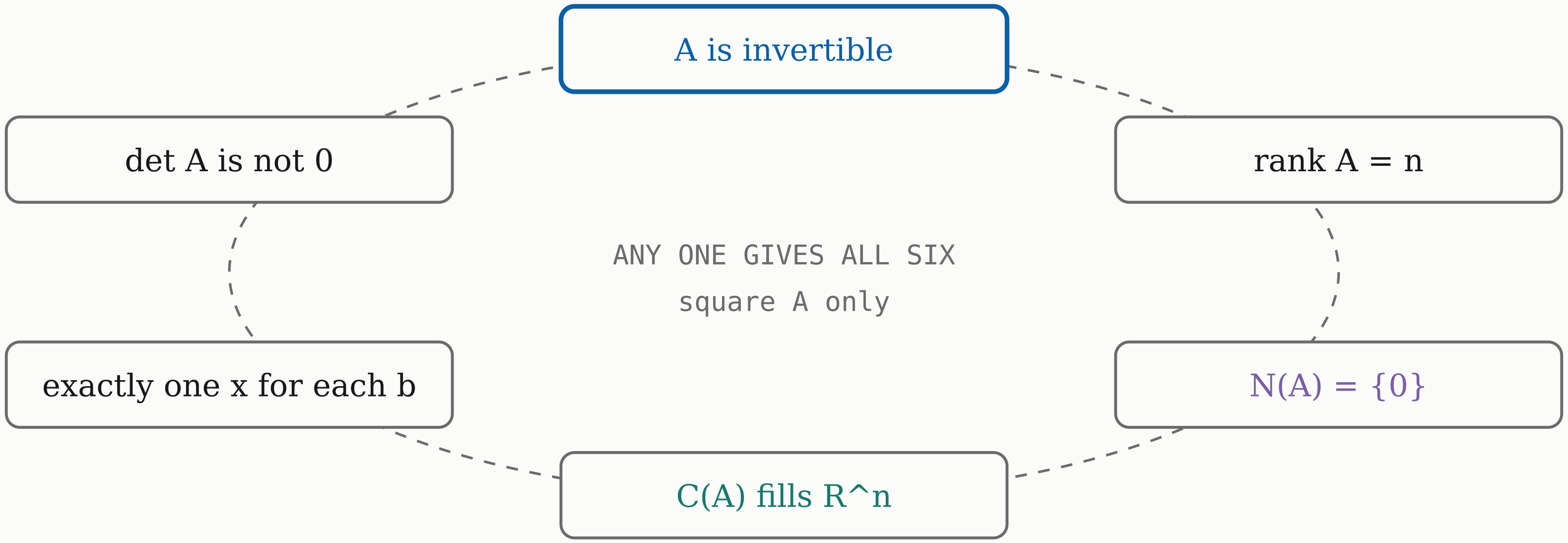
# Invertibility and Singular Matrices

SQUARE A

$$\boxed{\begin{aligned} A \text{ invertible} &\iff r = n \iff \mathcal{N}(A) = \{0\} \\ &\iff \text{columns independent} \iff \mathcal{C}(A) = \mathbb{R}^n \end{aligned}}$$

and $\iff Ax = b$ has one solution for every $b$

det a preview; the determinant is a later module




Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.63 invertibility, p. 83; 3.65, p. 84; 3.21, p. 62. Springer, 2024.

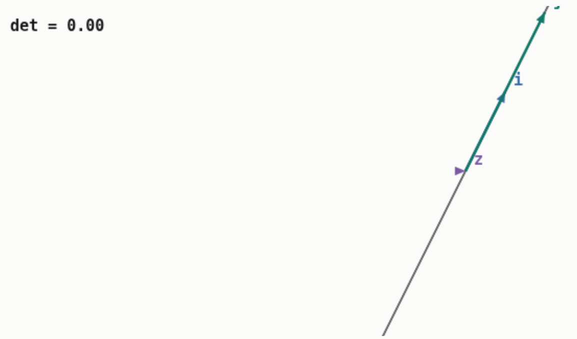
# Invertibility and Singular Matrices

SINGULAR $$\boxed{\mathcal{N}(B) \neq \{0\} \iff B \text{ singular}}$$

B square and rank deficient

**WORKED EXAMPLE**

$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has $r = 1$, and $B(2, -1)^\top = 0$: a nonzero input lands on zero.




The unit square flattens; the area, and the inverse, are gone.

Sanderson, G. ***Essence of Linear Algebra***, ch. 7. 3Blue1Brown, 2016 · Axler, 4th ed., 3.80, p. 91.

# Invertibility and Singular Matrices

PREVIEW · LEAST SQUARES $X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, \qquad X^\top X = \begin{bmatrix} 1+4+9 & 2+8+18 \\ 2+8+18 & 4+16+36 \end{bmatrix} = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$

the fit $\min_\beta \|X\beta - y\|^2 \;\Rightarrow\; X^\top X\beta = X^\top y$

entries $(X^\top X)_{ij}$ is column $i$ dotted with column $j$; the off-diagonal pair agree

Goodfellow, I., Bengio, Y. and Courville, A. ***Deep Learning***, ch. 2. MIT Press, 2016.

# Invertibility and Singular Matrices

SINGULAR $$X \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 4-4 \\ 6-6 \end{bmatrix} = 0 \implies X^\top X \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0$$

collinear column 2 is twice column 1: $(2,-1)^\top \in \mathcal{N}(X)$

NULL SPACES $$\boxed{X^\top X z = 0 \;\Rightarrow\; \|Xz\|^2 = z^\top X^\top X z = 0 \;\Rightarrow\; Xz = 0}$$

so $\mathcal{N}(X^\top X) = \mathcal{N}(X)$

the fit $\beta + t(2,-1)^\top$ gives the same $X\beta$: not identifiable

Goodfellow, I., Bengio, Y. and Courville, A. ***Deep Learning***, ch. 2. MIT Press, 2016.

# Invertibility and Singular Matrices

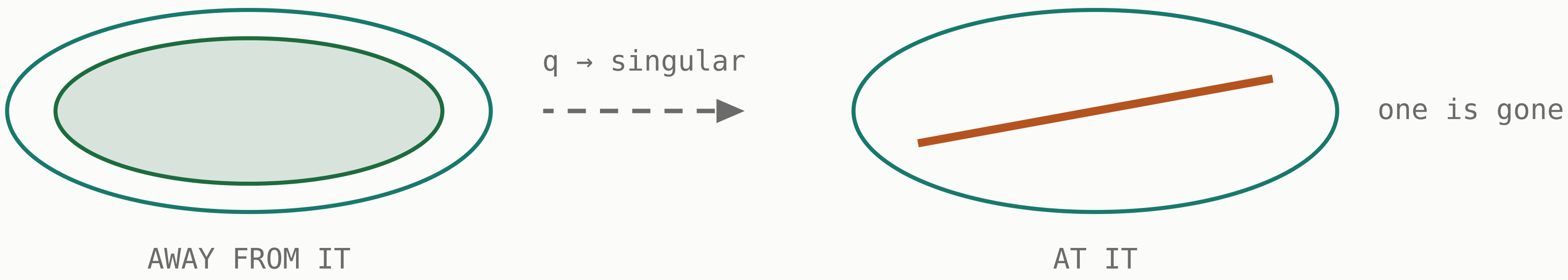
AT A SINGULARITY $\boxed{J(q) \text{ loses rank } \implies \text{ a tool direction is lost}}$

rank drop $\mathcal{C}(J)$ shrinks: some $\dot{x}$ has no $\dot{q}$

nearby joint rates grow without bound for tool directions along the lost one

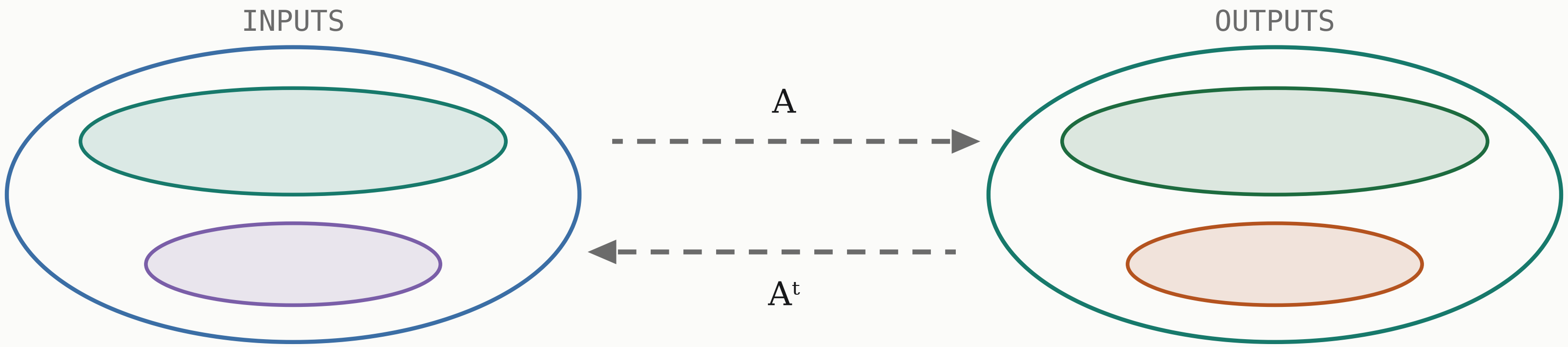
NEXT $\mathcal{C}(A)$, $\mathcal{N}(A)$, $\mathcal{C}(A^\top)$, $\mathcal{N}(A^\top)$

one matrix four spaces, one rank



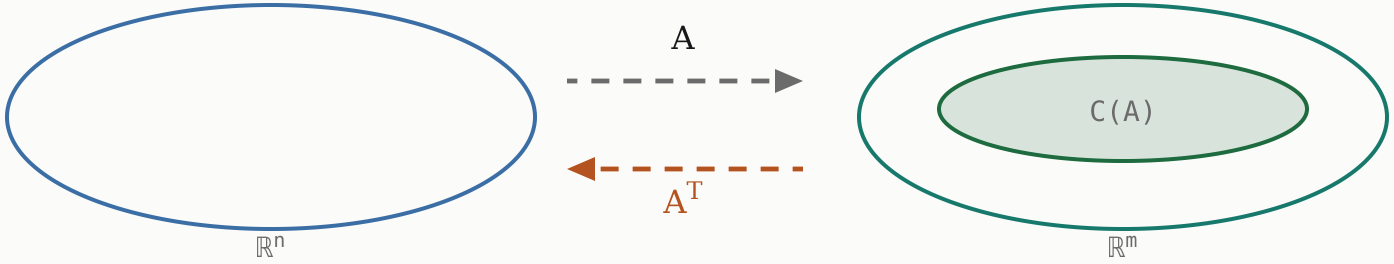

Lynch, K. M. and Park, F. C. ***Modern Robotics***, §5.3 singularity analysis, p. 191. Cambridge University Press, 2017 preprint.

# The Four Fundamental Subspaces



# The Four Fundamental Subspaces

SO FAR $A : \mathbb{R}^n \to \mathbb{R}^m, \qquad \mathcal{C}(A)$ and $\mathcal{N}(A)$




but $A^\top : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map too

SO ASK $\boxed{\text{what spaces come from } A^\top?}$

the answer $\mathcal{C}(A^\top)$ and $\mathcal{N}(A^\top)$: two more, for free

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.54 transpose, p. 77. Springer, 2024.

# The Four Fundamental Subspaces

NEW NAMES $\boxed{\mathrm{Row}(A) = \mathcal{C}(A^\top), \qquad \mathcal{N}(A^\top) = \{y : A^\top y = 0\}}$

$\mathbb{R}^n$ · INPUTS

row space $C(A^\top)$

null space $N(A)$

$\mathbb{R}^m$ · OUTPUTS

column space $C(A)$

left null $N(A^\top)$

**left null** $A^\top y = 0$ is $y^\top A = 0^\top$: y multiplies from the left

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.54 transpose, p. 77; 3.11 null space, p. 59. Springer, 2024.

# The Four Fundamental Subspaces

OUTPUT SIDE

$$\mathcal{C}(A) = \text{span}\{(1,0,2)^\mathsf{T}, (0,1,1)^\mathsf{T}\}$$

$\mathcal{N}(A^\mathsf{T})$ $= \text{span}\{(-2,-1,1)^\mathsf{T}\}$

INPUT SIDE

$$\text{Row}(A) = \text{span}\{(1,0,1)^\mathsf{T}, (0,1,1)^\mathsf{T}\}$$

$\mathcal{N}(A)$ $= \text{span}\{(-1,-1,1)^\mathsf{T}\}$

```
row 1   1 0 1
row 2   0 1 1    2r1 + r2 = r3
row 3   2 1 3    so −2r1 − r2 + r3 = 0
```

so $\boxed{y = (-2,-1,1)^\mathsf{T}, \qquad A^\mathsf{T} y = 0}$

read it the row weights

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.57 column rank equals row rank, p. 78. Springer, 2024.

# The Four Fundamental Subspaces

RANK EQUALITY $\boxed{\operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T}) = r}$

| r = 2 | m - r = 1 |
| --- | --- |

the output side: r = 2, m - r = 1

so column rank equals row rank

ALL FOUR

$$\begin{array}{ll} \dim \mathcal{C}(A) = r & \dim \mathcal{N}(A) = n - r \\ \dim \mathcal{C}(A^\mathsf{T}) = r & \dim \mathcal{N}(A^\mathsf{T}) = m - r \end{array}$$

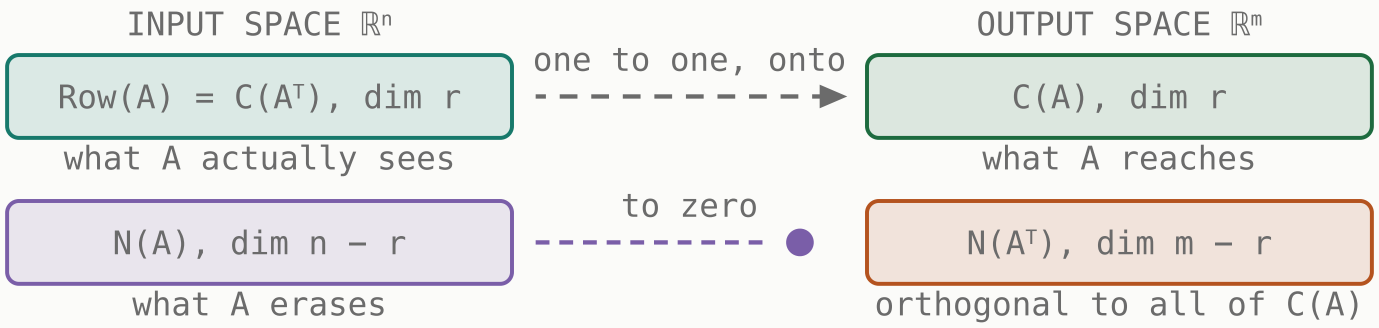
two sums $r + (n - r) = n$ and $r + (m - r) = m$

**WORKED EXAMPLE**

Our $A$: $r = 2$, so $2 + 1 = 3$ columns and $2 + 1 = 3$ rows.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.57 column rank equals row rank, p. 78; 3.21, p. 62. Springer, 2024.

# The Four Fundamental Subspaces




**M1 IN ONE LINE** $\boxed{\dim = r,\ n-r,\ r,\ m-r}$

later the two pairs form orthogonal-complement decompositions

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.21, p. 62; 3.57, p. 78. Springer, 2024.

SELECTED REFERENCES

# Courses, books and lectures


1. Strang, G. *18.06 Linear Algebra*, Lectures 3, 7 and 10. MIT OCW, 2010. CC BY-NC-SA 4.0.
2. Grizzle, J. W. *ROB 501: Mathematics for Robotics*, Michigan. CC BY-NC 4.0.
3. Axler, S. *Linear Algebra Done Right*, 4th ed., 2024. Open access.
4. Goodfellow, Bengio and Courville. *Deep Learning*, MIT Press, 2016, ch. 2.
5. Sanderson, G. *Essence of Linear Algebra*, ch. 2, 3 and 7. 3Blue1Brown, 2016.

SELECTED REFERENCES

# Papers and books


6. Strang, G. The fundamental theorem of linear algebra. *Amer. Math. Monthly*, 1993.
7. Black, K. et al. $\pi_0$. *RSS*, 2025; arXiv:2410.24164, posted 2024.
8. Lipman, Y. et al. Flow Matching for Generative Modeling. *ICLR*, 2023. arXiv:2210.02747.
9. Lynch and Park. *Modern Robotics*, §5.3, §6.3, 2017.
10. Barfoot, T. D. *State Estimation for Robotics*, ch. 4. Cambridge, 2017.