Linear Algebra II

Coordinates, Determinant, Eigenstructure

Mathematics for Robotics · ROB-GY 6013 · NYU Tandon

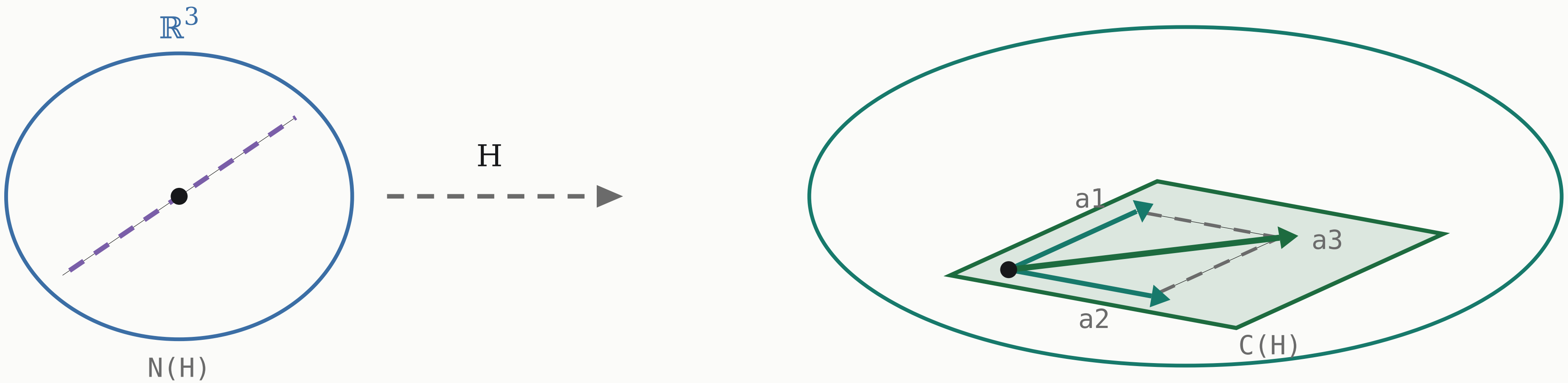
Jing Zhang · Fall 2026 · September 9

One Map, Many Rulers

$H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

RANK–NULLITY $\dim\ker H + \dim\operatorname{im} H = 1 + 2 = 3$

$\operatorname{im} H$ the plane the outputs fill



$$\boxed{\mathcal{N}(A) = \ker(A) = \{x \in \mathbb{R}^n : Ax = 0\}}$$

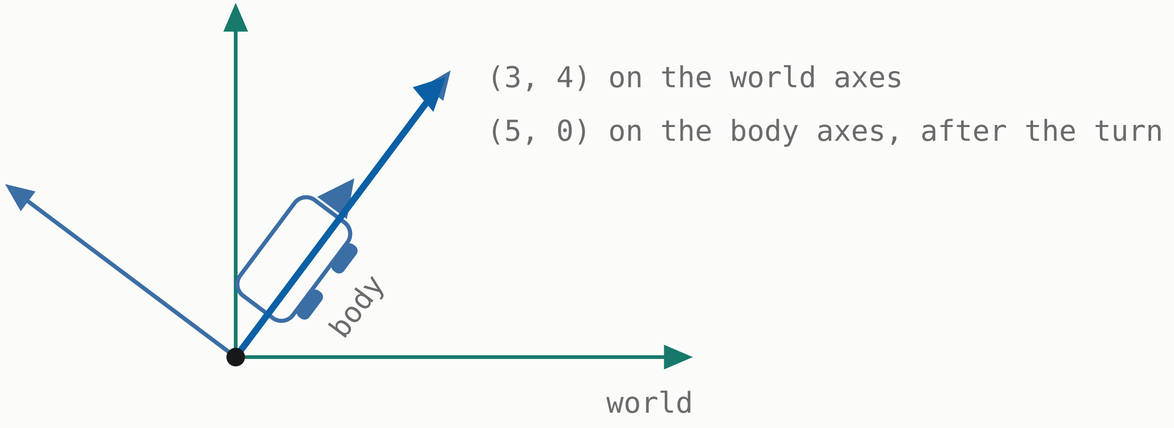
$$\boxed{\mathcal{N}(H) = \operatorname{span}\{[-1, -1, 1]^\mathsf{T}\}}$$

$\mathcal{N}(A) = \ker(A)$ the inputs A sends to 0

$x \in \mathbb{R}^n$ an input column of n reals

span all its multiples: a line

One Map, Many Rulers



Which choice makes the matrix simple?

One Map, Many Rulers

1 · COORDINATES

what are these numbers written in?

2 · ROTATIONS

which change of basis is a turn?

3 · DETERMINANT

does it collapse, and does it flip?

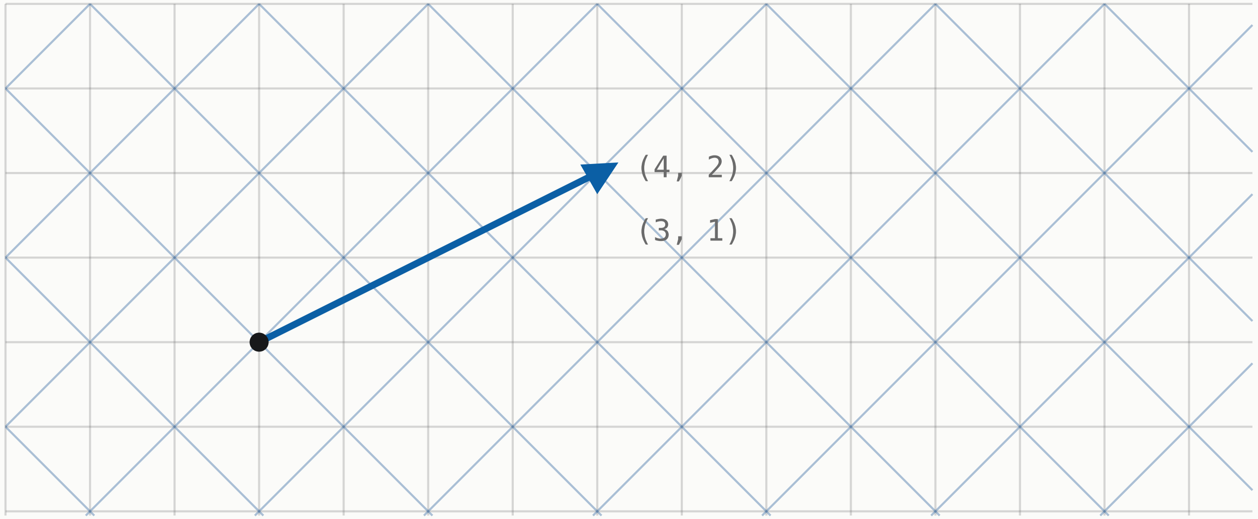
4 · EIGENVECTORS

which directions survive?

5 · DIAGONALIZATION

what is A to the k?

Coordinates and Change of Basis



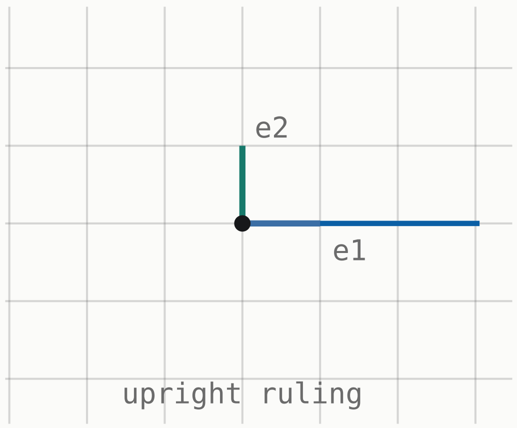
Coordinates and Change of Basis

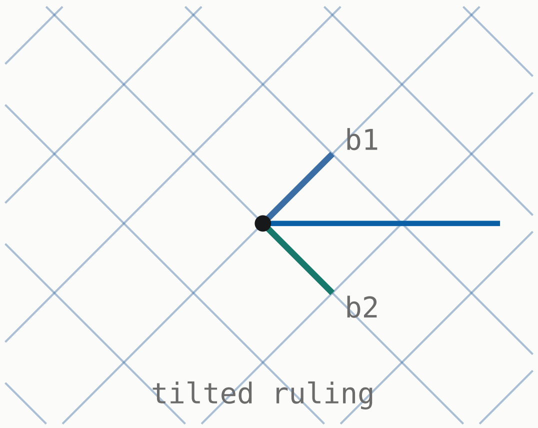
$$\mathcal{B} = (b_1, \ldots, b_n) \text{ is a basis of } \mathbb{R}^n \iff \text{independent and spans } \mathbb{R}^n$$

`independent` none of them is a mixture of the rest

`spans` every vector is some mixture of the list

`n` a basis of $\mathbb{R}^n$ has exactly n vectors





Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.26 basis, p. 39. Springer, 2024. Open access.

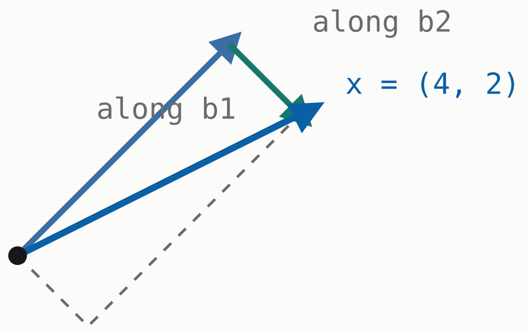
Coordinates and Change of Basis

ONE RECIPE ONLY $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$ in exactly one way

α_j the amount of b_j the recipe calls for

exists because the list spans

only one because the list is independent



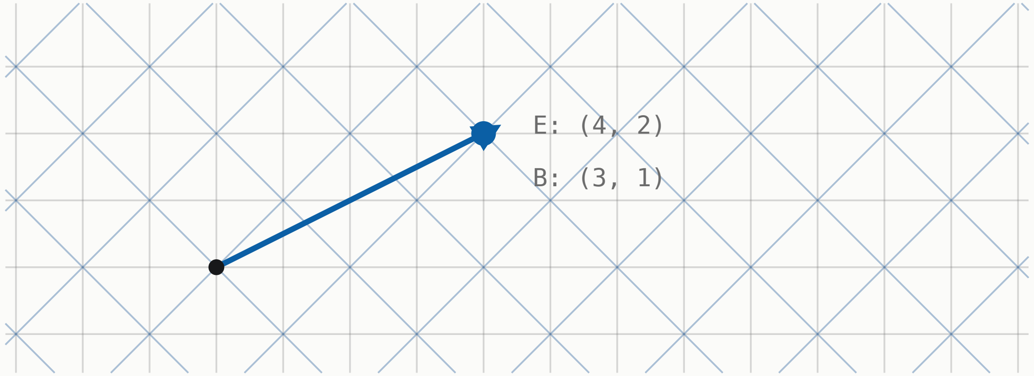
Axler, S. ***Linear Algebra Done Right***, 4th ed., 2.28 criterion for basis, p. 39. Springer, 2024. Open access.

Coordinates and Change of Basis

COORDINATE VECTOR

$$[x]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ where } x = \alpha_1 b_1 + \cdots + \alpha_n b_n$$

- x — the arrow itself, out in the world
- $[x]_{\mathcal{B}}$ — a vector in $\mathbb{R}^n$ holding the coefficients, not the arrow x
- α_j — how much of b_j the one recipe uses



Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.73 the matrix of a vector, p. 88. Springer, 2024. Open access.

Coordinates and Change of Basis

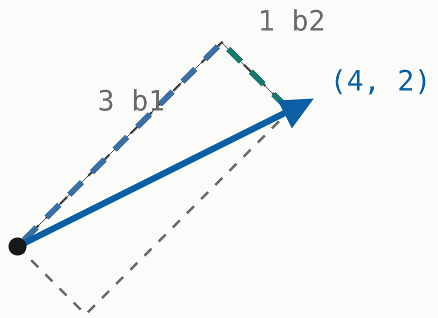
WORKED EXAMPLE

$[x]_\mathcal{E} = (4, 2)^\mathsf{T}$, $b_1 = (1, 1)$, $b_2 = (1, -1)$. Then $\alpha_1 + \alpha_2 = 4$ and $\alpha_1 - \alpha_2 = 2$, so $\alpha_1 = 3$, $\alpha_2 = 1$. Check: $3b_1 + b_2 = (4, 2)$.

$$[x]_\mathcal{E} = (4, 2)^\mathsf{T} \longrightarrow [x]_\mathcal{B} = (3, 1)^\mathsf{T}$$

$[x]_\mathcal{E}$ the given standard-coordinate column

$[x]_\mathcal{B}$ the same vector in $\mathcal{B}$: three b_1, one b_2

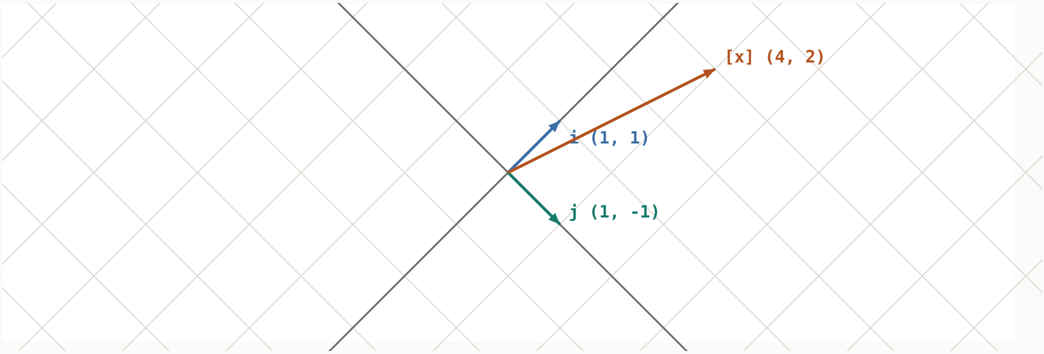


Coordinates and Change of Basis

CONVERT TO STANDARD COORDINATES $\boxed{[x]_{\mathcal{E}} = B\,[x]_{\mathcal{B}}}$, $\qquad B = \big[\,[b_1]_{\mathcal{E}}\ \cdots\ [b_n]_{\mathcal{E}}\,\big]$

B columns are the basis vectors in standard coordinates

$[x]_{\mathcal{E}}$ written x once the standard basis is understood



$$\begin{array}{ccccc} 3 \cdot & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & + 1 \cdot & \begin{bmatrix} 1 \\ -1 \end{bmatrix} = & \begin{bmatrix} 4 \\ 2 \end{bmatrix} \\ & \text{b1} & & \text{b2} & \text{[x] in E} \end{array}$$

Coordinates and Change of Basis

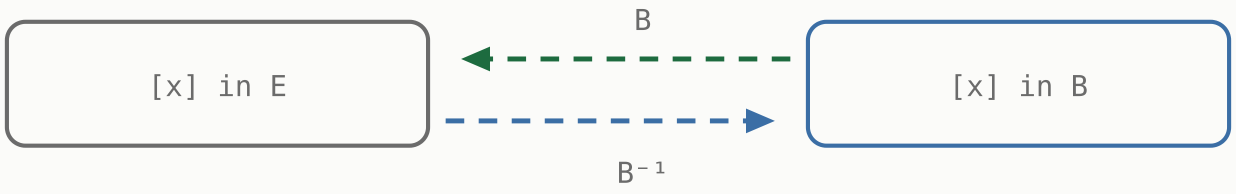
CONVERT BACK WITH THE INVERSE $\boxed{[x]_{\mathcal{B}} = B^{-1}[x]_{\mathcal{E}}}$

B^{-1} undoes B, back to $\mathcal{B}$-coordinates

WORKED EXAMPLE

$M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$: swap, negate, divide by $\det M$.

$B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\det B = -2$, so $B^{-1} = \frac{1}{2}B$ and $[x]_{\mathcal{B}} = \frac{1}{2}(6,2)^{\mathsf{T}} = (3,1)^{\mathsf{T}}$.



Coordinates and Change of Basis

$$A^{-1}A = AA^{-1} = I$$

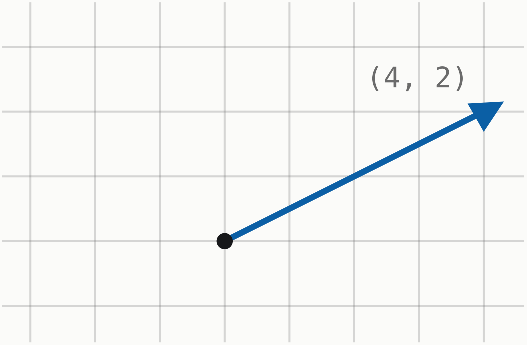
STANDARD BASIS

$$e_1, \ldots, e_n : \quad B = I \;\Rightarrow\; [x]_{\mathcal{E}} = I^{-1}x = x$$

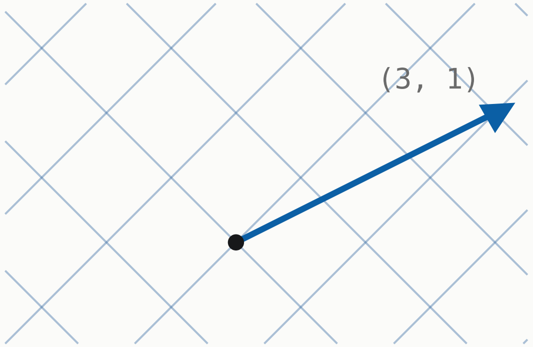
A^{-1} undoes A from either side

e_j 1 in slot j, 0 everywhere else

I its basis matrix, and its own inverse



B = I



B = [b1 b2]

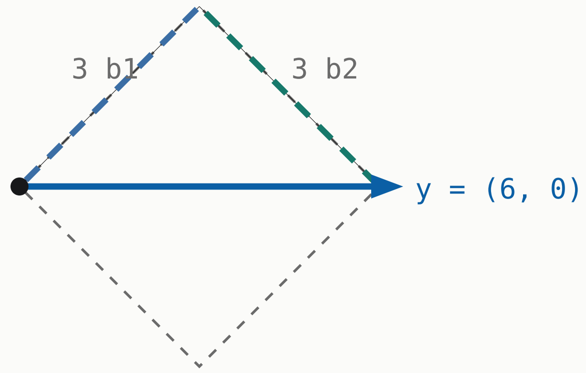
PRACTICE

Coordinates and Change of Basis

Same $\mathcal{B} = \{(1,1),(1,-1)\}$. What is $[y]_{\mathcal{B}}$ for $y = (6,0)$?

$[y]_{\mathcal{B}} = (3,3)^{\mathsf{T}}$

Check it forwards: $3(1,1) + 3(1,-1) = (6,0)$. Or read it off: $B^{-1}(6,0) = \frac{1}{2}(6,6) = (3,3)$.

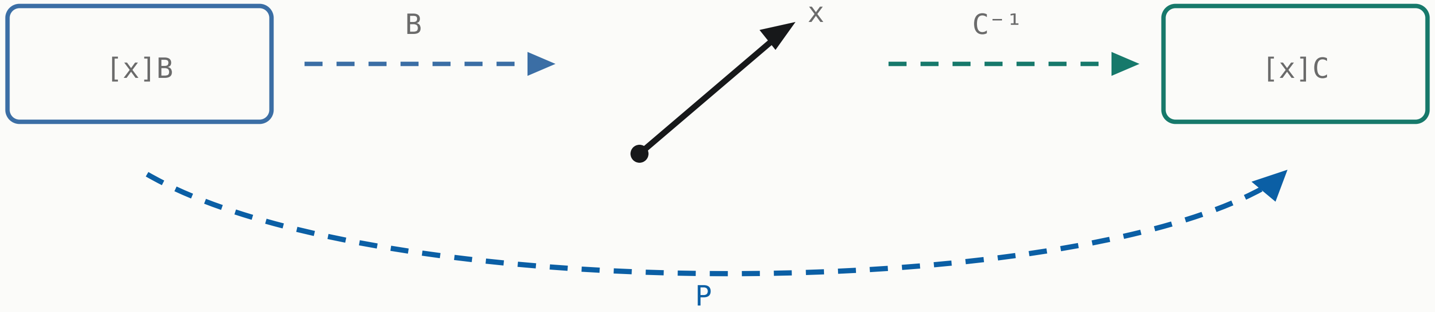


Coordinates and Change of Basis

$$[x]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[x]_{\mathcal{B}}, \qquad P_{\mathcal{C}\leftarrow\mathcal{B}} = C^{-1}B$$

P turns B-coordinates into C-coordinates

$C^{-1}B$ rebuild the vector from B, then read it off in C



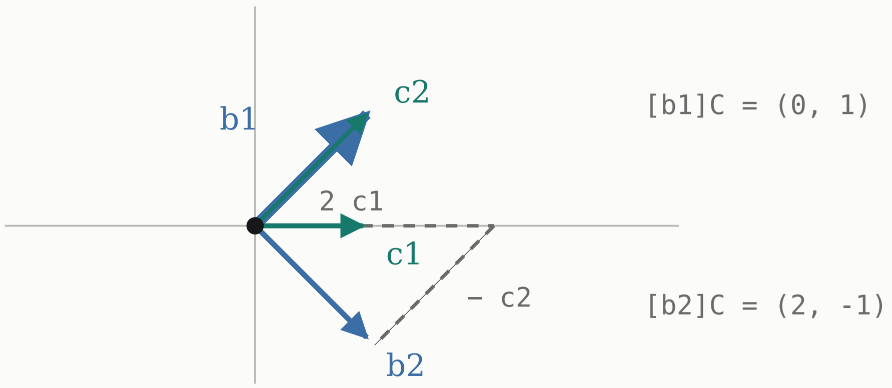
Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.84 change-of-basis formula, p. 93. Springer, 2024. Axler writes this matrix as M(I, …), the identity map read between two bases.

Coordinates and Change of Basis

WORKED EXAMPLE

$\mathcal{C} = \{(1,0),(1,1)\}$, so $C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $P = C^{-1}B = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$.

$b_1 = 0c_1 + 1c_2$ and $b_2 = 2c_1 - 1c_2$: the columns of P.



Coordinates and Change of Basis

$$x = (4,2) \text{ in } \mathcal{E}, \qquad [x]_{\mathcal{B}} = (3,1), \qquad [x]_{\mathcal{C}} = (2,2)$$

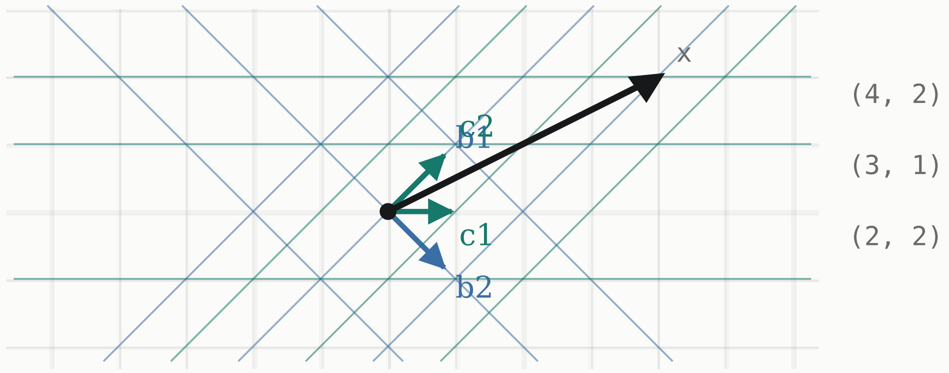
`(4,2)` read in the standard basis E

`(3,1)` read in B = {(1,1), (1,-1)}

`(2,2)` read in C = {(1,0), (1,1)}

WORKED EXAMPLE

$$P\,[x]_{\mathcal{B}} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



$P = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

Coordinates and Change of Basis

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = B^{-1}C$$

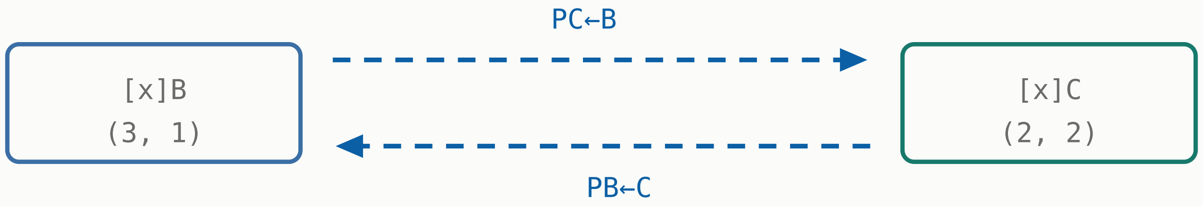
`left side` C-coordinates in, B-coordinates out

`B⁻¹C` rebuild from C, then read off in B

$$P^{-1} = \begin{pmatrix} 1/2 & 1 \\ 1/2 & 0 \end{pmatrix}, \qquad P^{-1}(2,2) = (3,1)$$

`(2,2)` the C-column

`(3,1)` back to the B-column

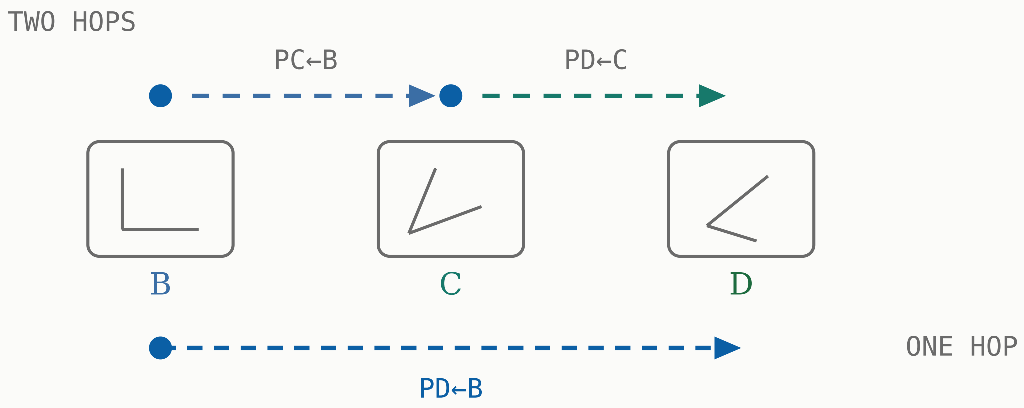


Coordinates and Change of Basis

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\, P_{\mathcal{C}\leftarrow\mathcal{B}}$$

`inner pair` C meets C and cancels

`order` the right factor acts first

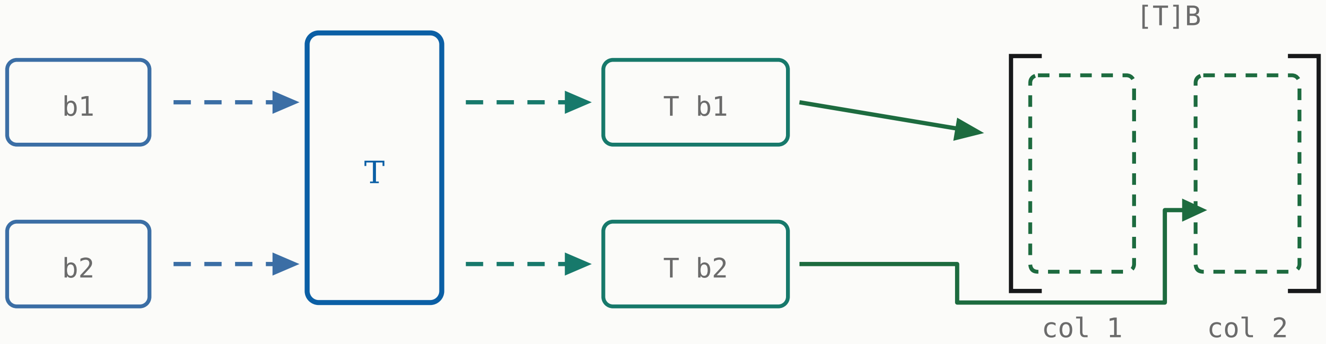


Strang, G. ***18.06 Linear Algebra***, Lecture 31, change of basis. MIT OpenCourseWare, 2010.

Coordinates and Change of Basis

$$[Tx]_{\mathcal{B}} = [T]_{\mathcal{B}}\,[x]_{\mathcal{B}}, \qquad \text{column } j = [Tb_j]_{\mathcal{B}}$$

T	a linear map from the space to itself
[x]_B	coordinates in, coordinates out, same basis
column j	feed in b_j, write the answer in the same basis
[T]_B	depends on T and on the ordered basis B



Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.31 matrix of a linear map M(T), p. 69. Springer, 2024.

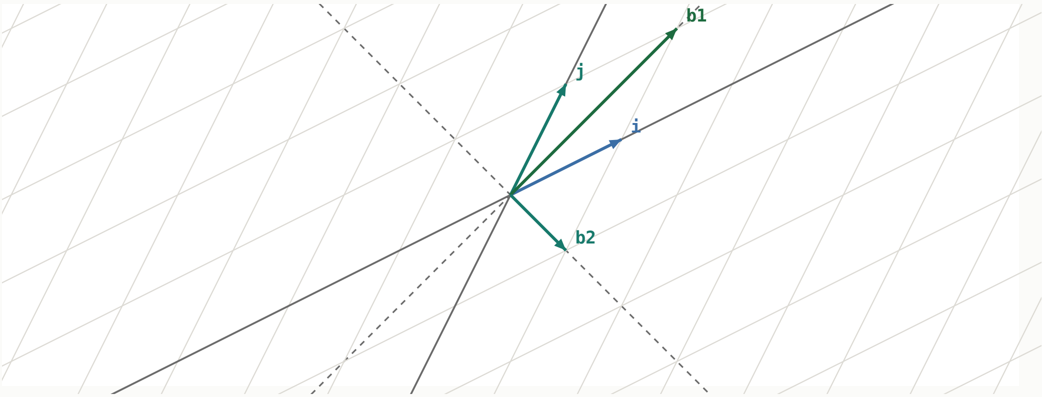
Coordinates and Change of Basis

WORKED EXAMPLE

$Ab_1 = (3,3) = 3b_1 + 0b_2$, so column 1 is $(3,0)^\mathsf{T}$. $Ab_2 = (1,-1) = 0b_1 + 1b_2$, so column 2 is $(0,1)^\mathsf{T}$.

$$\boxed{[A]_{\mathcal{B}} = B^{-1}AB = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}$$

`B⁻¹AB`	rebuild, apply A, read off
`3 and 1`	the factor each basis direction is stretched by
`off-diagonal`	zero here: the two directions no longer mix



Coordinates and Change of Basis

$$[T]_{\mathcal{C}} = P\,[T]_{\mathcal{B}}\,P^{-1}, \qquad P = P_{\mathcal{C}\leftarrow\mathcal{B}}$$

P	puts B-coordinates into C, the same P as before
[T]_B	the operator acts, in B
P⁻¹	takes the C-column into B, and acts first
similar	the name for two matrices related this way

$$\begin{array}{ccc} [x]_C & \xrightarrow{[T]_C} & [Tx]_C \\ \downarrow{\scriptstyle P^{-1}} & & \uparrow{\scriptstyle P} \\ [x]_B & \xrightarrow{[T]_B} & [Tx]_B \end{array}$$

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.84 change of basis, p. 93. Springer, 2024. The word ***similar*** is Strang's: MIT 18.06, Lecture 28.

$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Coordinates and Change of Basis

WORKED EXAMPLE

$C^{-1}A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$, then $(C^{-1}A)\,C = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = [A]_{\mathcal{C}}$.

$$\boxed{\operatorname{tr}(A) = \textstyle\sum_i a_{ii}}$$

$$\boxed{\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(A)}$$

`tr` sum of the diagonal entries

$B^{-1}AB$ new basis, same sum

A

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

2 + 2 = 4

[A]C

$$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

1 + 3 = 4

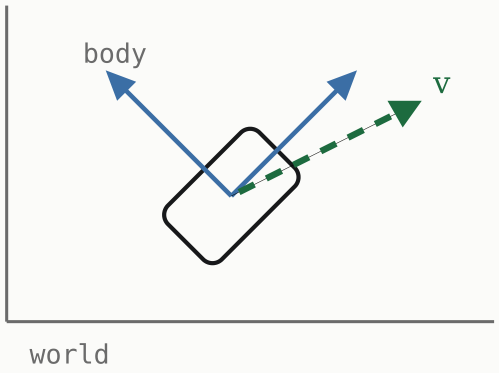
[A]B

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

3 + 1 = 4

Coordinates and Change of Basis

The same gains, written in two frames



IN THE WORLD FRAME IN THE BODY FRAME

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{bmatrix} \dashrightarrow \begin{bmatrix} 0.9 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Axler, S. ***Linear Algebra Done Right***, 4th ed., 3.84 change-of-basis formula, p. 93. Springer, 2024. Open access.

PRACTICE

Coordinates and Change of Basis

$[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ with $\mathcal{B} = \{(1,1),(1,-1)\}$. What is $[T]_{\mathcal{E}}$?

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\mathrm{B}\,(\cdot)\,\mathrm{B}^{-1}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

IN B   IN E

$$[x]_B = B^{-1}[x]_E$$

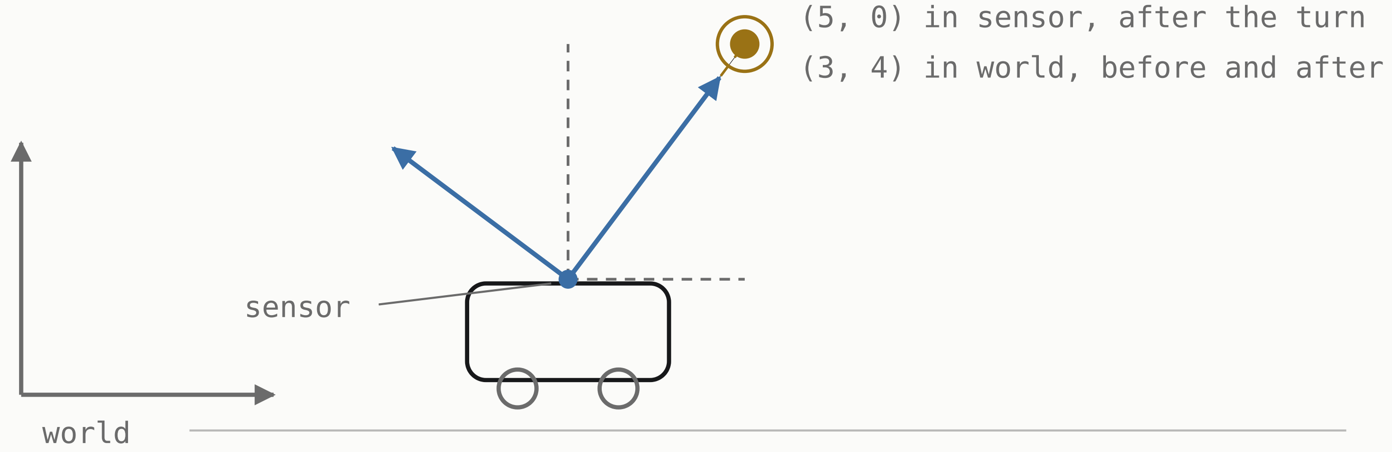
$$[Tx]_B = [T]_B[x]_B$$

$$[Tx]_E = B[Tx]_B$$

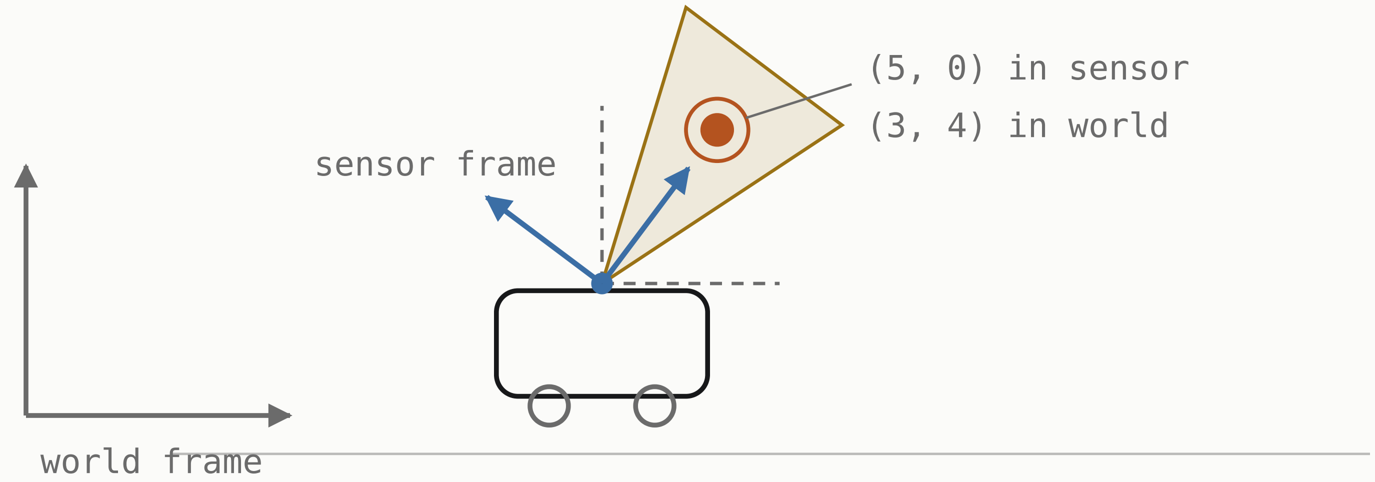
$$\boxed{[T]_E = B[T]_B B^{-1}}$$

the three lines into B, act, back out; together they are the box

Rotations and Frames



Rotations and Frames



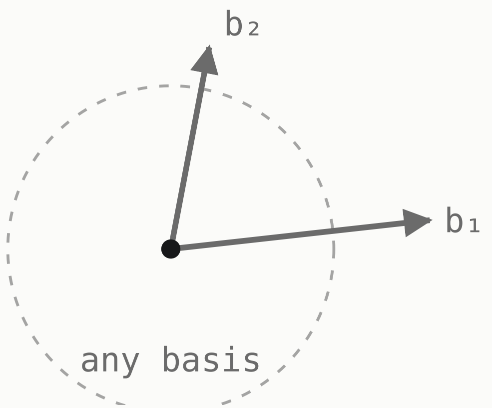
Rotations and Frames

ORTHONORMAL $$\langle q_i, q_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

q_i the basis vectors, one for each axis

$i = j$ every one of them has length one

$i \neq j$ every pair meets at a right angle

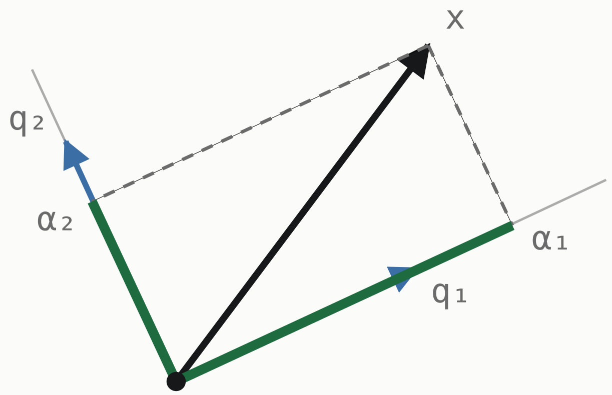


Axler, S. ***Linear Algebra Done Right***, 4th ed., 6.22 orthonormal, p. 197; 6.27 orthonormal basis, p. 199. Springer, 2024. Open access.

Rotations and Frames

$$[x]_{\mathcal{Q}} = Q^{\mathsf{T}} x, \qquad \text{i.e. } \alpha_j = \langle x, q_j \rangle$$

Q	the $n \times n$ matrix with columns $q_1, \ldots, q_n$
Q^{T}	exactly Q^{-1}: a transpose, nothing to solve
α_j	how far x reaches along axis j



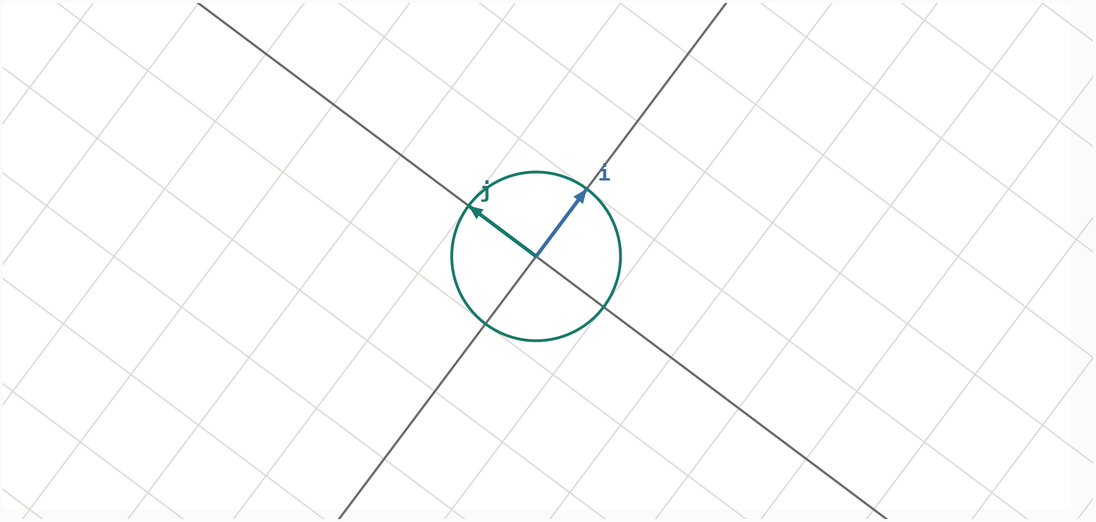
Axler, S. ***Linear Algebra Done Right***, 4th ed., 6.30(a) writing a vector as a linear combination of an orthonormal basis, p. 200. Springer, 2024. Open access.

Rotations and Frames

ORTHOGONAL $Q^\top Q = I \iff$ the columns of Q are orthonormal, $\quad Q^{-1} = Q^\top$

$(Q^\top Q)_{ij}$	column i dotted with column j
I	ones on the diagonal, zeros off it
$Q^{-1} = Q^\top$	the inverse costs one transpose

$$Q^\top Q \quad \begin{array}{c c} & \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} & \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \end{array} = I$$



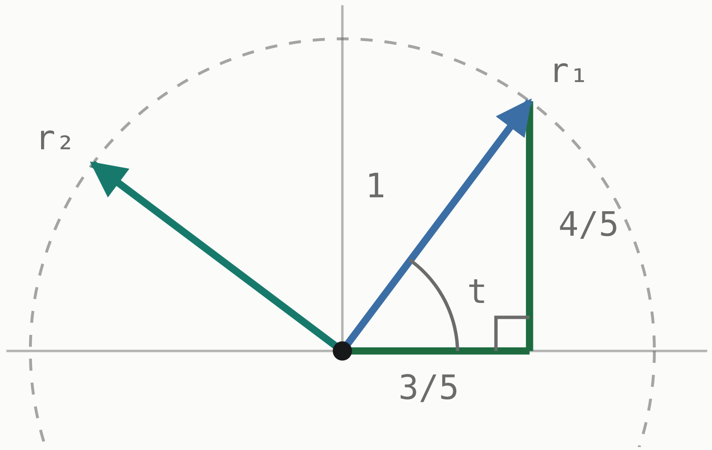
Both ends are orthogonal; the frames between them are smaller.

Grizzle, J. W. ***ROB 501: Mathematics for Robotics***, notes 11, symmetric and orthogonal matrices. University of Michigan, 2018. CC BY-NC 4.0.

Rotations and Frames

WORKED EXAMPLE

$R = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$. Column one with itself: $\frac{9}{25} + \frac{16}{25} = 1$. Column one with column two: $\frac{-12}{25} + \frac{12}{25} = 0$. So $R^{\mathsf{T}}R = I$, with $\cos t = \frac{3}{5}$, $\sin t = \frac{4}{5}$, $t \approx 53.13°$.



Rotations and Frames

ROTATION GROUP $SO(n) = \{R \in \mathbb{R}^{n\times n} \;:\; R^\top R = I,\ \det R = 1\}$

$R^\top R = I$ lengths and angles survive the map

$\det R = 1$ a turn, not a mirror; the number itself arrives next chapter

n 2 for planar orientations, 3 for spatial ones

	$R^\top R = I$	det R
rotation	✓	
reflection	✓	
scaling × 2	✗	

→ ch. 3

Lynch, K. M. & Park, F. C. ***Modern Robotics***, Definitions 3.1 and 3.2, p. 70. Cambridge, 2017. SO(3) and SO(2) are defined separately there; the general n is our shorthand.

$$R = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

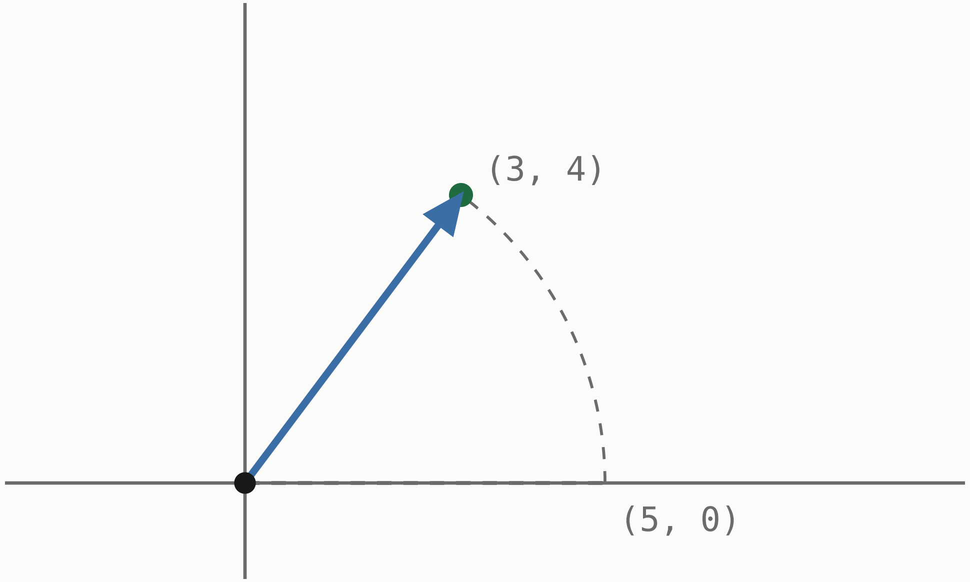
Rotations and Frames

ACTIVE $x' = Rx: \qquad R\,(5,0) = (3,4)$

x the arrow before the turn

R the turn itself, the same for every arrow

x' a different arrow, of the same length



Rotations and Frames

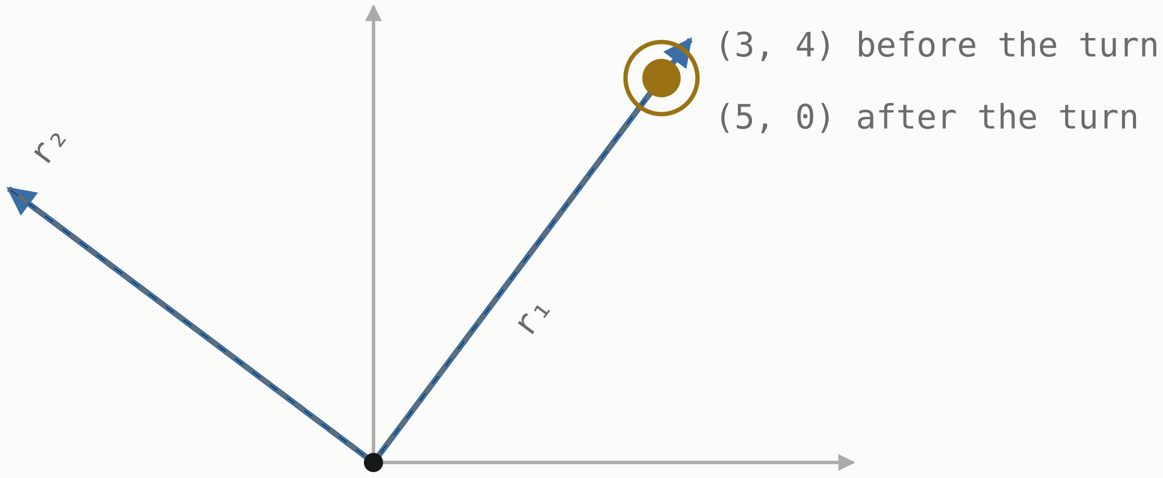
$$R = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

PASSIVE $[x]_{\mathcal{R}} = R^{\mathsf{T}} x: \qquad R^{\mathsf{T}}(3,4) = (5,0)$

x one physical point; nothing about it moves

R^{T} chapter 1's $B^{-1}x$, with $B = R$

$[x]_{\mathcal{R}}$ the same point's new address



Lynch, K. M. & Park, F. C. ***Modern Robotics***, §3.2.1.2, uses of rotation matrices, p. 71. Cambridge University Press, 2017.

Rotations and Frames

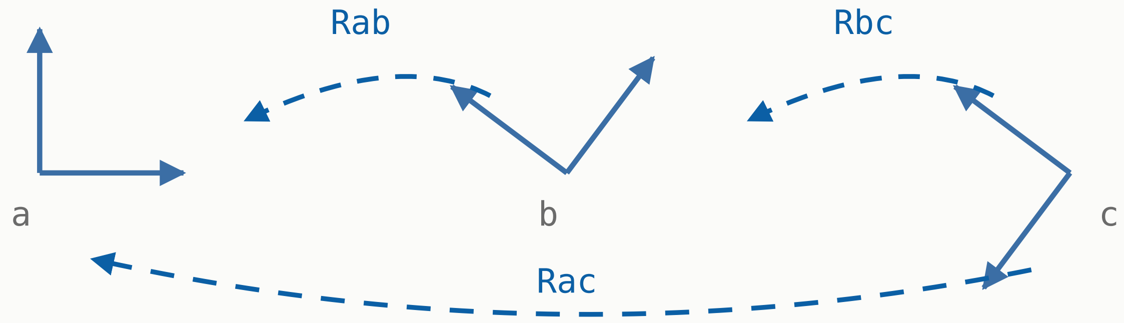
COMPOSE $R_{ac} = R_{ab} R_{bc}$

R_{ab} b-coordinates rewritten as a-coordinates

`the inner b` cancels; read the product right to left

WORKED EXAMPLE

$R_{ab} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, $R_{bc} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, so $R_{ac} = \frac{1}{5}\begin{bmatrix} -4 & -3 \\ 3 & -4 \end{bmatrix}$ — still unit, still perpendicular.

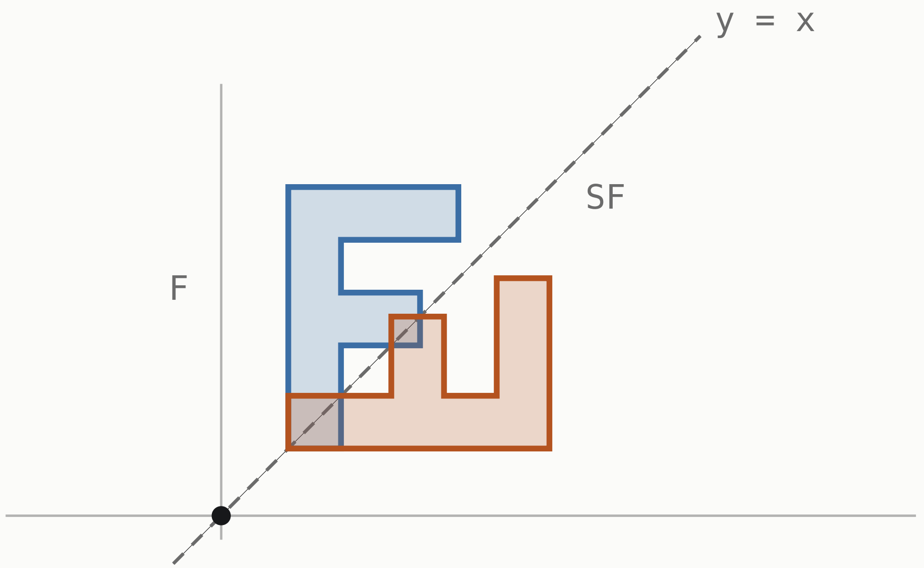


Rotations and Frames

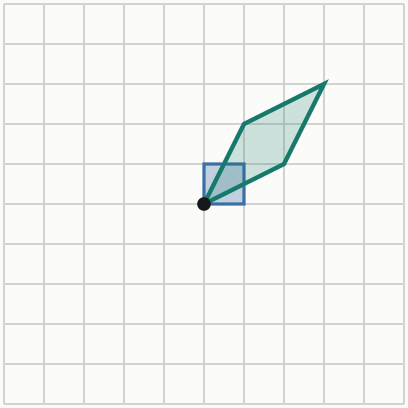
$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad S^{\mathsf{T}} S = I$$

S the mirror in the line $y = x$

$S^{\mathsf{T}} S = I$ every length and every angle survives



The Determinant



THE AREA CHANGES

OR COLLAPSES

The Determinant

What number tells a turn from a mirror?

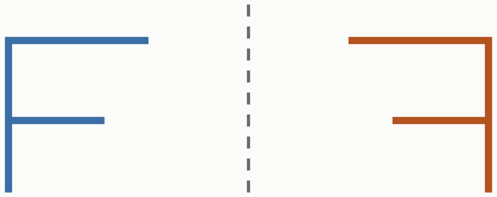
R

$$\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

S

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

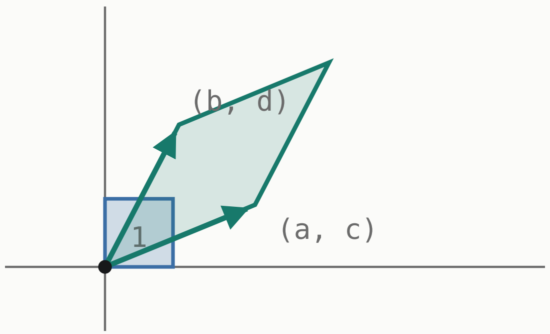
EVERY LENGTH KEPT, EVERY ANGLE KEPT



The Determinant

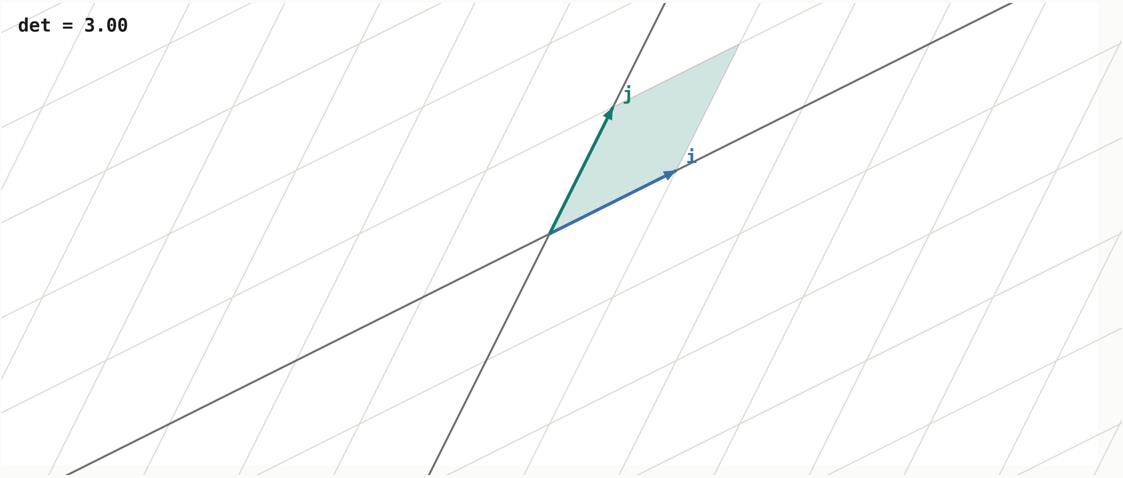
DEFINITION $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

`a, b, c, d`	the four entries, columns first
`ad − bc`	the signed area of the parallelogram the columns span
`signed`	the sign records orientation, not size



Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.43 determinant of a matrix, p. 355; 9.47 the 2×2 formula, p. 356. Springer, 2024. Open access.

The Determinant



WORKED EXAMPLE

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: $ad - bc = 2 \cdot 2 - 1 \cdot 1 = 3$, so every area in the plane is tripled.

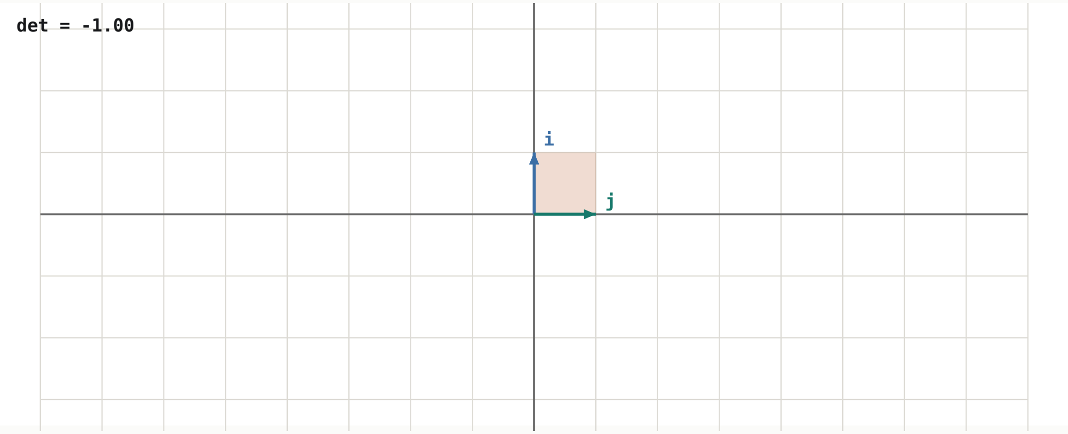
The Determinant

$$\det S = 0 \cdot 0 - 1 \cdot 1 = -1$$

$0 \cdot 0$ the diagonal product

$1 \cdot 1$ the off-diagonal product

-1 the plane has been turned over: a right-handed pair of axes is now left-handed



The square flattens, then reopens the other way round.

Sanderson, G. ***Essence of Linear Algebra***, ch. 6: the determinant. 3Blue1Brown, 2016.

The Determinant

$$\det R = \frac{3}{5} \cdot \frac{3}{5} - \left(-\frac{4}{5}\right) \cdot \frac{4}{5} = \frac{9}{25} + \frac{16}{25} = 1$$

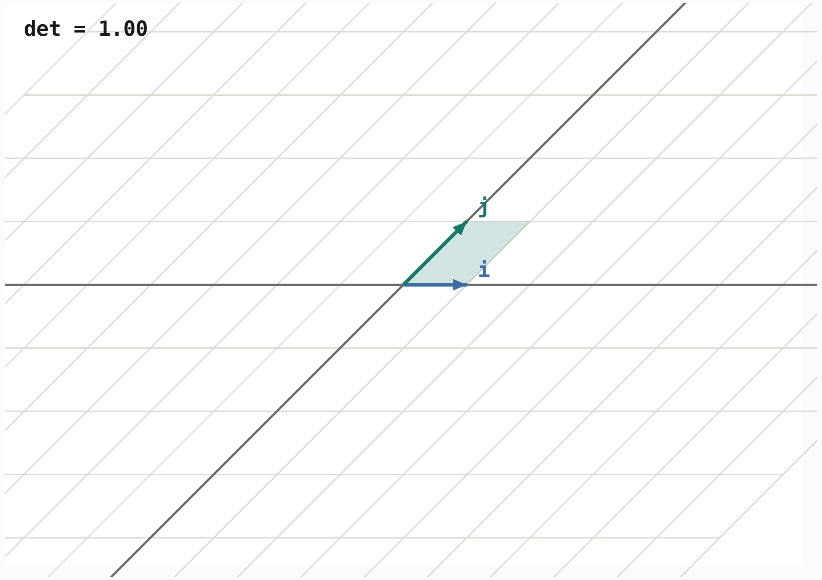
9/25 the diagonal product
+16/25 minus a negative, so the off-diagonal term adds
= 1 no stretch, no flip: a turn

MAP	$Q^{\top}Q = I$	DET	VERDICT
rotation R	✓	+1	SO(2)
mirror S	✓	−1	reflection

Lynch, K. M. & Park, F. C. ***Modern Robotics***, Definitions 3.1 and 3.2, p. 70. Cambridge, 2017.

The Determinant

RULE	WHY, IN VOLUME
$\det(AB) = \det A \cdot \det B$	do one map, then the other
$\det A^{\mathsf{T}} = \det A$	transposing leaves every volume alone
$\det A^{-1} = 1/\det A$	undoing a scaling by 3 scales by 1/3
$\det(cA) = c^n \det A$	all n directions scaled by c
triangular T: $\det T = t_{11} \cdots t_{nn}$	shearing moves the box, not its volume



Row five, not row one: the other four rules are not drawn.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.48 upper-triangular, p. 356; 9.49 multiplicative, p. 357; 9.56(a) transpose, p. 360. Springer, 2024. Open access.

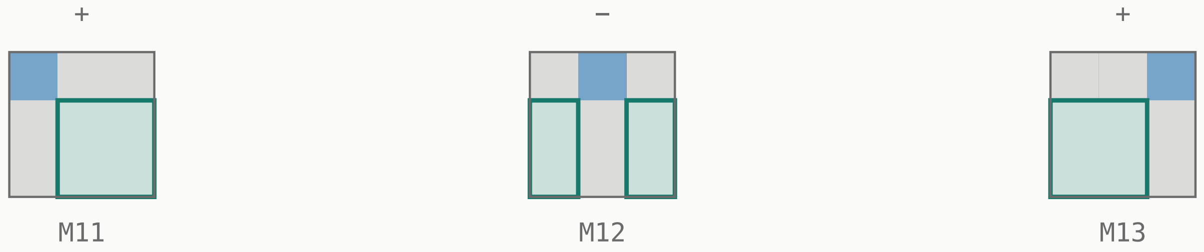
The Determinant

$$\det A = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

a_{1j} the three entries of the first row

M_{1j} the 2×2 determinant left when row 1 and column j are deleted

$+ \; - \; +$ a chequerboard of signs, starting at plus



Strang, G. ***18.06 Linear Algebra***, Lecture 19: determinant formulas and cofactors. MIT OpenCourseWare, 2010. CC BY-NC-SA 4.0.

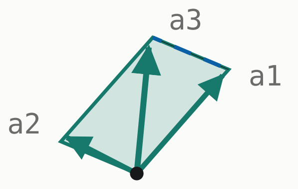
The Determinant

$H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

WORKED EXAMPLE

$\det H = 1(1 \cdot 3 - 1 \cdot 1) - 0(0 \cdot 3 - 1 \cdot 2) + 1(0 \cdot 1 - 1 \cdot 2) = 2 - 0 - 2 = 0$

Last week $a_3 = a_1 + a_2$ by eye; one number says the same thing.

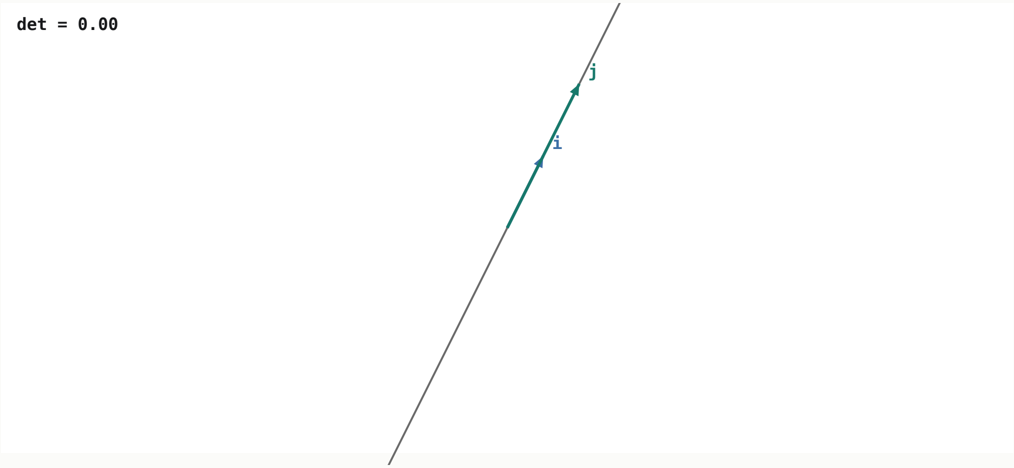


NO THICKNESS

The Determinant

$$\det A = 0 \iff A \text{ is singular} \iff \mathcal{N}(A) \neq \{0\} \iff \operatorname{rank} A < n$$

`det A = 0`	the unit box is flattened
`singular`	no inverse exists
$\mathcal{N}(A)$	the null space: some nonzero input is sent to zero
`rank`	fewer than n independent columns



Every point of the plane lands on one line through the origin.

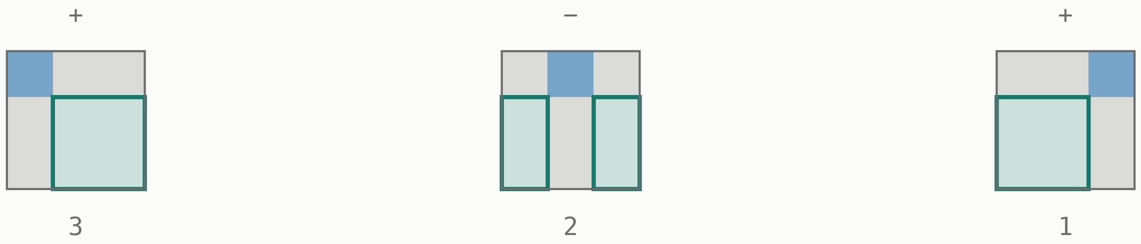
Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.50 invertible if and only if nonzero determinant, p. 357. Springer, 2024. Open access.

PRACTICE

The Determinant

$$K = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \text{ singular?}$$

`singular` no inverse: some direction is destroyed

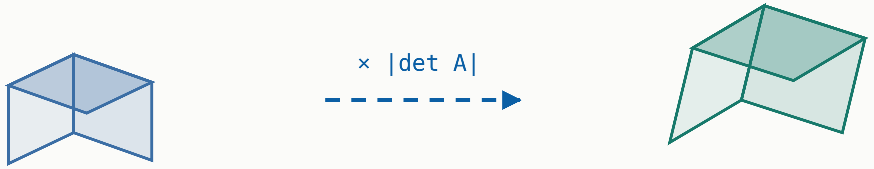


$\det K = 4$, so no. $2(4-1) - 1(2-0) + 0 = 6 - 2 = 4$, so $Kx = b$ has exactly one solution for every b. Or eliminate — subtracting a multiple of one row leaves $\det$ alone: $r_2 - \frac{1}{2} r_1$, then $r_3 - \frac{2}{3} r_2$, leaves pivots $2, \frac{3}{2}, \frac{4}{3}$, product 4.

The Determinant

$$\operatorname{vol} A(\Omega) = |\det A| \cdot \operatorname{vol} \Omega$$

Ω	any region you like
$A(\Omega)$	its image under A
$\lvert\det A\rvert$	one factor, the same for every region
`absolute value`	volume has no sign; orientation does

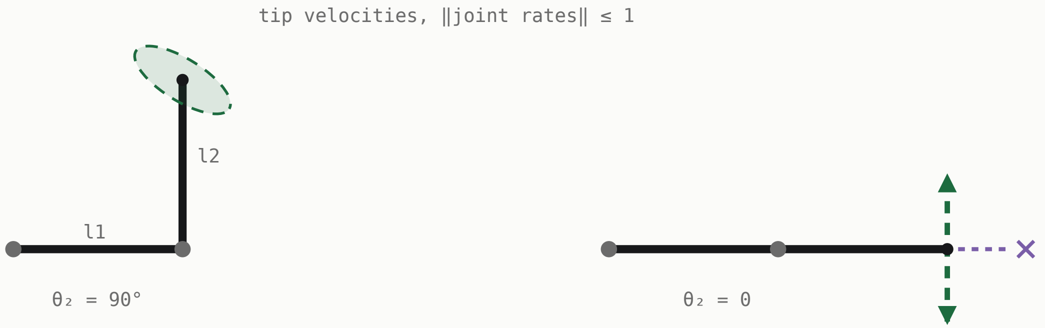


Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.61 T changes volume by a factor of |det T|, p. 363. Springer, 2024. Open access.

The Determinant

$$\det J(\theta) = l_1 l_2 \sin\theta_2$$

J	joint rates to tip velocity, M4's Jacobian
l_1, l_2	the two link lengths, both fixed
θ_2	the elbow angle
= 0	exactly when the elbow is straight or folded



WORKED EXAMPLE

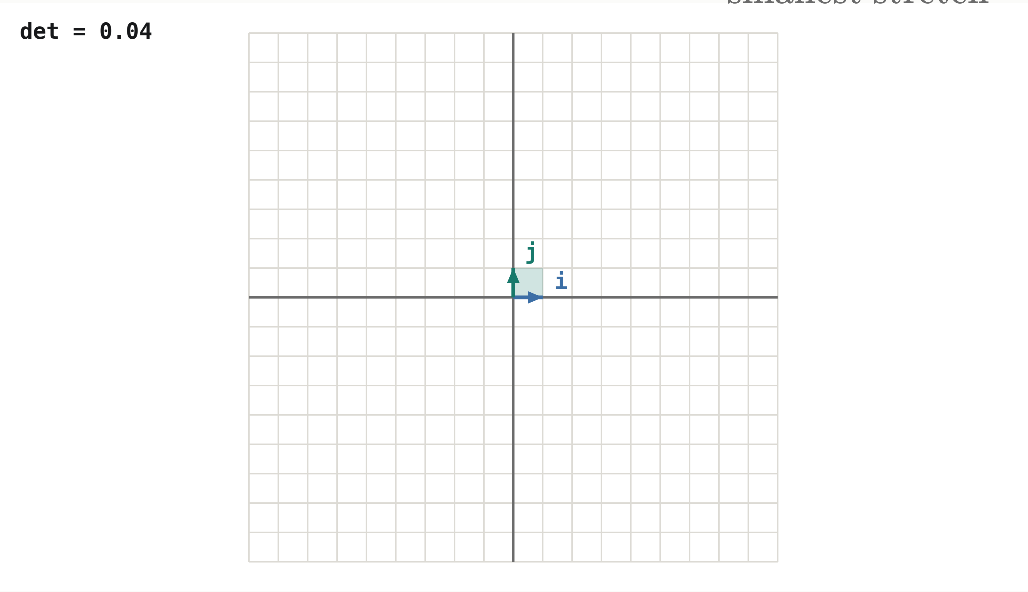
$l_1 = l_2 = 1$: $\det J = \sin\theta_2$, and that set has area $\pi|\sin\theta_2|$ — π at 90°, and 0 when the elbow is straight.

Lynch, K. M. & Park, F. C. ***Modern Robotics***, §5.3 singularities. Cambridge University Press, 2017.

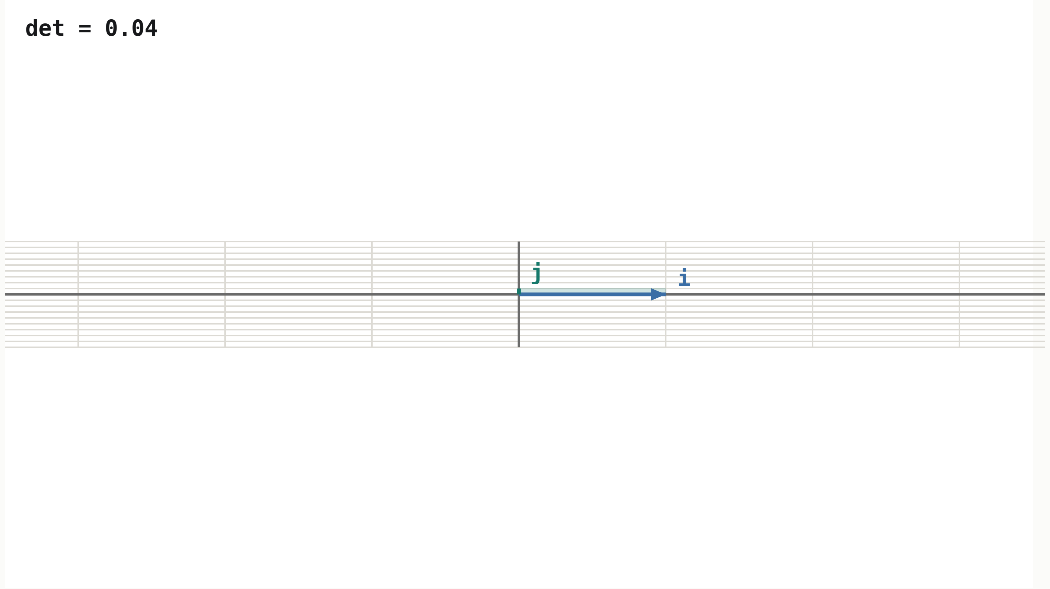
The Determinant

$$\det(0.01\, I_{10}) = (0.01)^{10} = 10^{-20}$$

$0.01\, I_{10}$	one hundredth of the identity in ten dimensions
$(0.01)^{10}$	one factor from each diagonal entry
10^{-20}	small, and every direction still scaled alike
κ	$\dfrac{\text{largest stretch}}{\text{smallest stretch}}$, the number to ask for — M4



$0.2\, I$: $\kappa = 1$



$\mathrm{diag}(1, 0.04)$: $\kappa = 25$

The Determinant

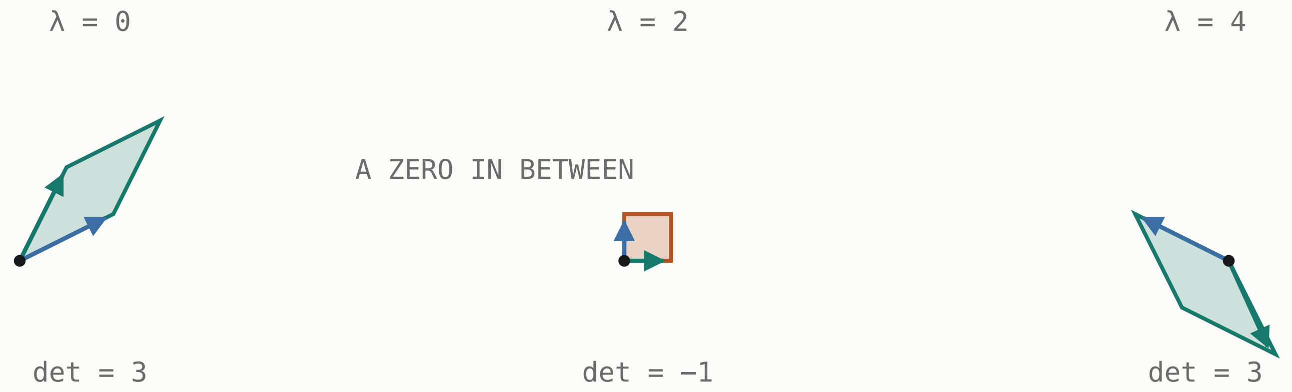
$$\det(A - \lambda I) = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}: \quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$$

$A - \lambda I$ A with λ subtracted along the diagonal

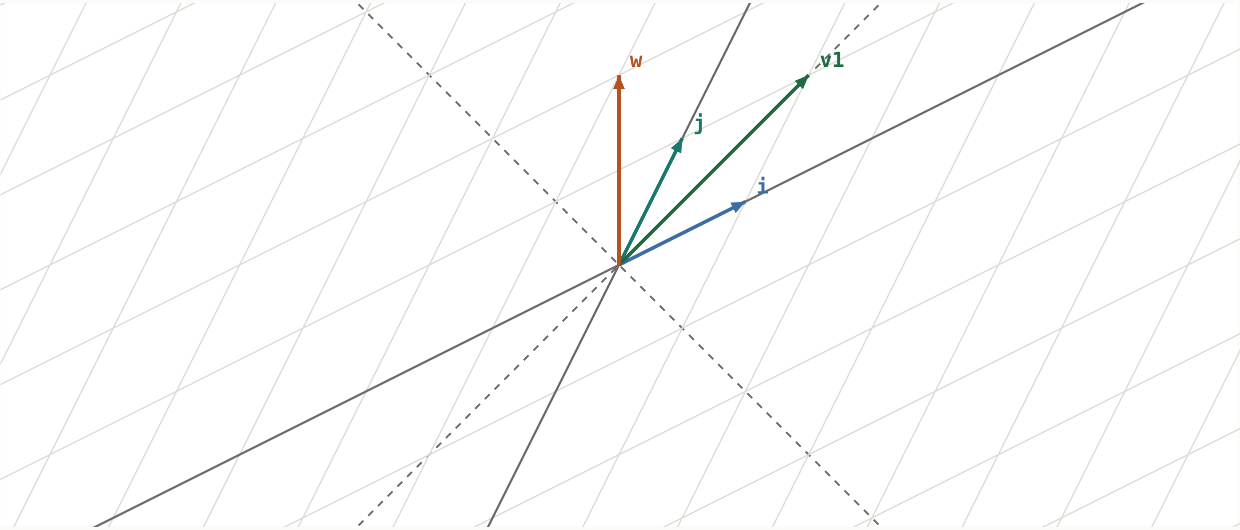
`det = 0` this chapter's test for collapse

λ unknown; the whole of the next chapter



Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.51, p. 358; 9.63 characteristic polynomial, p. 363. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

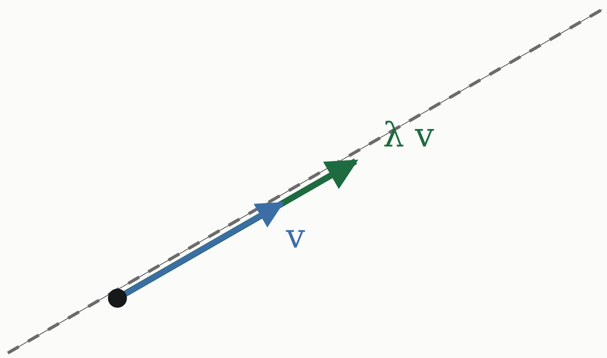


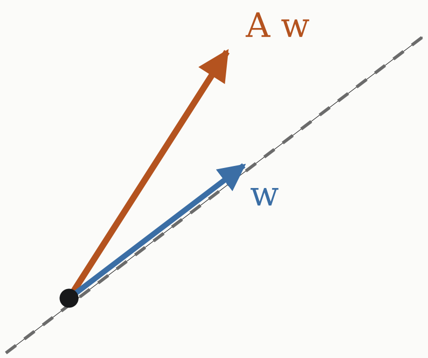
One rider keeps its dashed line. The other does not.

Eigenvalues and Eigenvectors

EIGENVECTOR $$Av = \lambda v, \qquad v \neq 0$$

A the map: a square matrix
v the eigenvector, a direction A does not turn
λ the eigenvalue, the factor v is stretched by
$v \neq 0$ required; $\lambda = 0$ is allowed





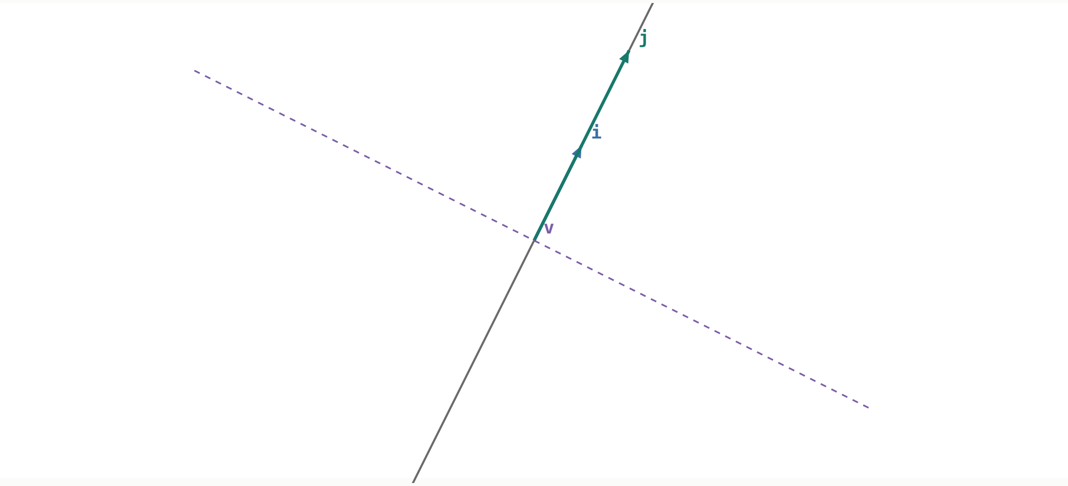
Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.5 eigenvalue, p. 134; 5.8 eigenvector, p. 135. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

Allow v = 0 and every number is an eigenvalue of every matrix.

$$\lambda = 0 \text{ is an eigenvalue} \iff \mathcal{N}(A) \neq \{0\} \iff \det A = 0$$

$\lambda = 0$	$Av = 0$ for some $v \neq 0$
$\mathcal{N}(A)$	M1's null space: what the map destroys
$\det A = 0$	chapter 3's test for a collapse



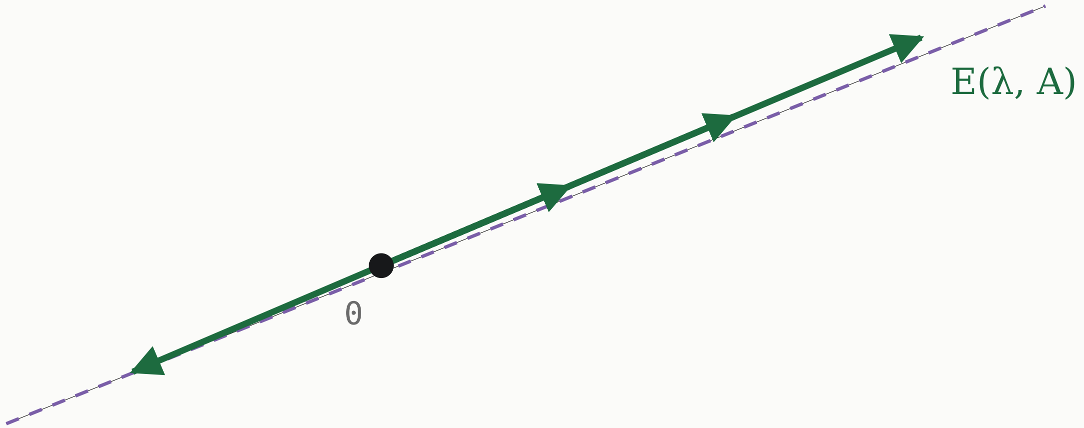
The marked arrow goes to the origin.

Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.50 invertible if and only if nonzero determinant, p. 357. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

EIGENSPACE $E(\lambda, A) = \mathcal{N}(A - \lambda I) = \{x : \ Ax = \lambda x\}$

$E(\lambda, A)$	every vector A stretches by λ, and 0
$\mathcal{N}(A - \lambda I)$	M1's null space, of a different matrix
`closed`	$A(\alpha x + \beta y) = \lambda(\alpha x + \beta y)$, so it is a subspace



Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.52 eigenspace, p. 164. Springer, 2024. Open access.

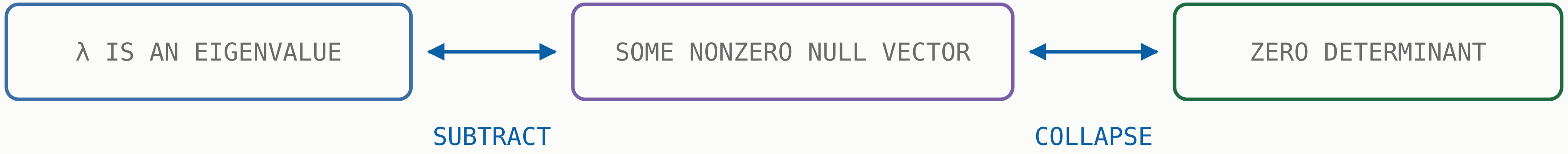
Eigenvalues and Eigenvectors

$$\lambda \text{ is an eigenvalue} \iff \exists\, v \neq 0 :\ (A - \lambda I)v = 0 \iff \det(A - \lambda I) = 0$$

`subtract` $\lambda v = \lambda I v$, so both terms move to one side

$\exists v \neq 0$ some nonzero v solves it, not the v you name

`collapse` chapter 3: it has one exactly when its determinant is 0



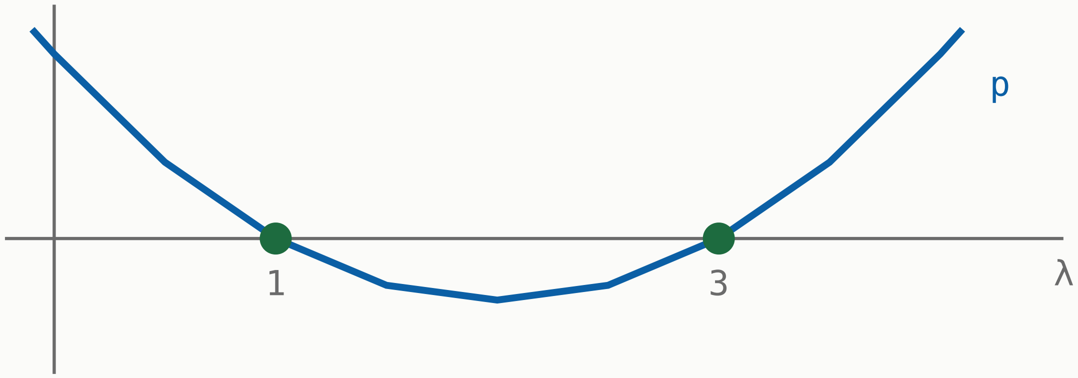
Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.7 equivalent conditions, p. 135; 9.51 eigenvalue test, p. 358. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

CHARACTERISTIC POLYNOMIAL $$p(\lambda) = \det(A - \lambda I), \qquad \underbrace{p = \lambda^2 - (\operatorname{tr} A)\lambda + \det A}_{2\times 2 \text{ only}}$$

p degree n in λ; its roots are the eigenvalues

$\operatorname{tr} A$ the sum of the diagonal entries



Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.63 characteristic polynomial, p. 363; 8.47 trace of a matrix, p. 326. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

WORKED EXAMPLE

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$$

expand $(2-\lambda)^2 = 4 - 4\lambda + \lambda^2$, then subtract 1

roots $\lambda = 3$ and $\lambda = 1$

check tr A = 4 and det A = 3 give the same polynomial

$$\begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix}$$

MAIN

minus

ANTI

Eigenvalues and Eigenvectors

WORKED EXAMPLE

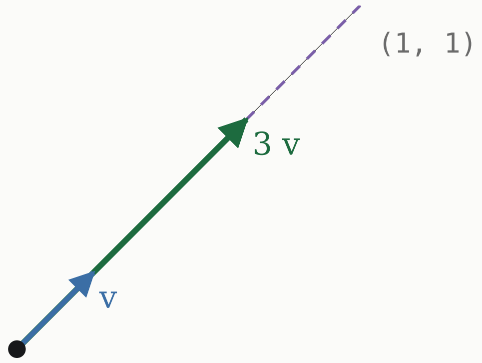
$\lambda = 3$: solve $(A - 3I)v = 0$, last week's null-space computation.

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

reduce $r_2 \to r_2 + r_1$, then $-r_1$: one pivot, one free

read $v_1 - v_2 = 0$, so $v_2 = t$ and $v = t(1,1)$

check $A(1,1) = (2+1,\, 1+2) = (3,3) = 3(1,1)$



Strang, G. ***18.06 Linear Algebra***, Lecture 21: eigenvalues and eigenvectors. MIT OCW, 2010. CC BY-NC-SA 4.0.

Eigenvalues and Eigenvectors

WORKED EXAMPLE

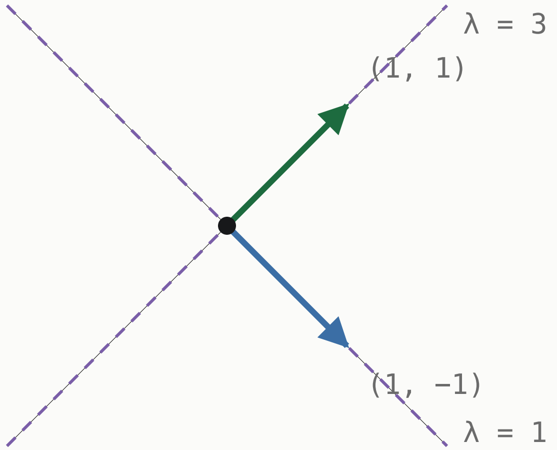
$\lambda = 1$: solve $(A - I)v = 0$.

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

reduce $r_2 \to r_2 - r_1$: one pivot, one free

read $v_1 + v_2 = 0$, so $v = t(1, -1)$

check $A(1, -1) = (2 - 1, 1 - 2) = (1, -1)$, unchanged



Eigenvalues and Eigenvectors

$$B^{-1}AB = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Ab_1 = 3b_1, \quad Ab_2 = 1\,b_2$$

$b_1 = (1, 1)$	column 1 of B, and the first diagonal entry is 3
$b_2 = (1, -1)$	column 2 of B, and the second diagonal entry is 1
`order`	swap the columns of B and the diagonal swaps with them

CHAPTER 1

THE BASIS THAT DIAGONALISED A

=

THIS CHAPTER

THE DIRECTIONS A DOES NOT TURN

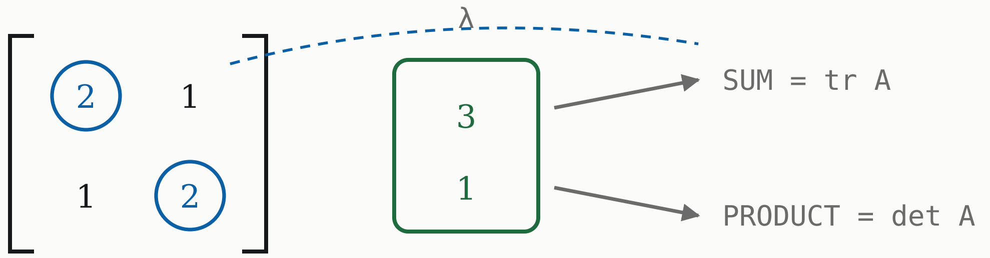
Eigenvalues and Eigenvectors

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n, \qquad \det A = \lambda_1 \cdots \lambda_n$$

$\operatorname{tr} A$	the sum of the diagonal entries of A
λ_j	the n roots of p, over $\mathbb{C}$ and with multiplicity
`over` $\mathbb{C}$	required: the next page is real and has no real root

WORKED EXAMPLE

$\operatorname{tr} A = 2 + 2 = 4 = 3 + 1$ and $\det A = 4 - 1 = 3 = 3 \cdot 1$.

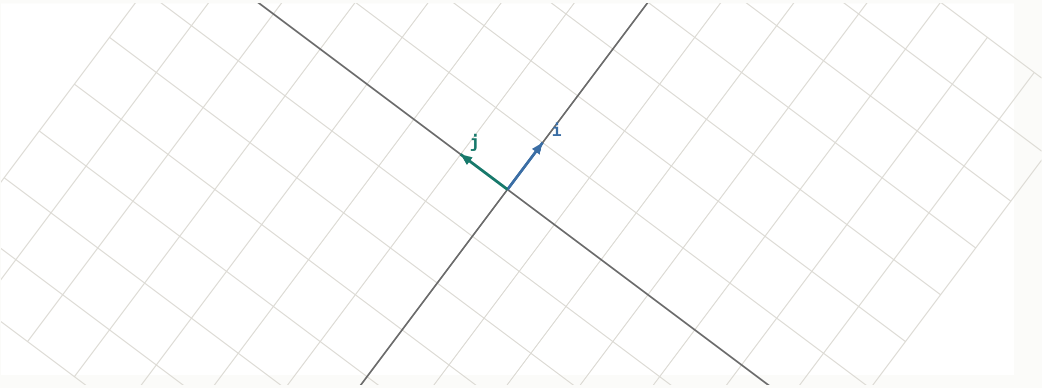


Axler, S. ***Linear Algebra Done Right***, 4th ed., 8.52 trace is the sum of the eigenvalues, p. 328; 9.55 the determinant is their product, p. 359. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

$$R = \tfrac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}, \quad p(\lambda) = \lambda^2 - \tfrac{6}{5}\lambda + 1, \quad \Delta = \tfrac{36}{25} - 4 < 0$$

$\operatorname{tr} R = 6/5$	$3/5 + 3/5$
$\det R = 1$	a rotation preserves area
$\Delta = b^2 - 4ac$	of $a\lambda^2 + b\lambda + c$; negative here, so no real root and no real direction survives



No dashed line appears.

Lynch, K. M. & Park, F. C. ***Modern Robotics***, Definition 3.2, SO(2), p. 70. Cambridge University Press, 2017.

PRACTICE

Eigenvalues and Eigenvectors

What are the eigenvalues of $\begin{pmatrix} 3 & 7 \\ 0 & 5 \end{pmatrix}$?

3 and 5: $\det(A - \lambda I) = (3-\lambda)(5-\lambda)$, and a triangular determinant is the product of the diagonal.

For $\lambda = 3$, $v = (1, 0)$. For $\lambda = 5$ it is $(7, 2)$, not $(0, 1)$: $(A - 5I)(7, 2) = (-14 + 14,\ 0) = 0$.

$$\begin{bmatrix} 3 & 7 \\ 0 & 5 \end{bmatrix}$$

λ = 3, 5

v = (1, 0) and (7, 2)

Axler, S. ***Linear Algebra Done Right***, 4th ed., 9.48 determinant of an upper-triangular matrix, p. 356. Springer, 2024. Open access.

Eigenvalues and Eigenvectors

SIMILAR MATRICES $Av = \lambda v \implies (P^{-1}AP)(P^{-1}v) = \lambda\,(P^{-1}v)$

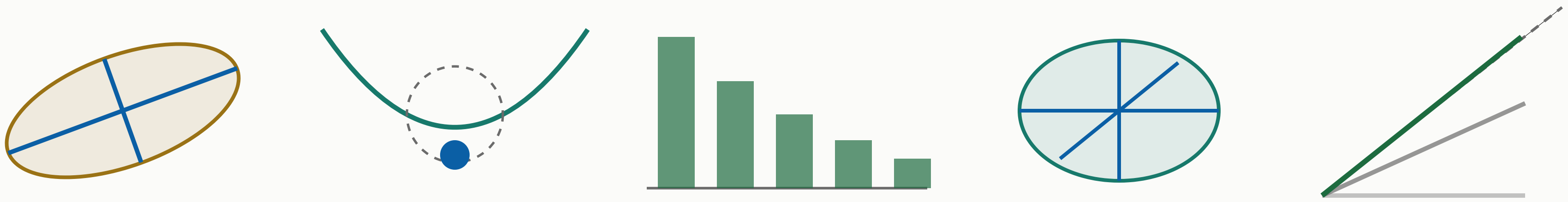
P	any change of basis
$P^{-1}v$	the same eigenvector, read in the new coordinates
λ	untouched: similar matrices share their eigenvalues

A	C⁻¹AC	B⁻¹AB
$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$	$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
{3, 1}	{3, 1}	{3, 1}

Strang, G. ***18.06 Linear Algebra***, Lecture 28: similar matrices and Jordan form. MIT OCW, 2010.

Eigenvalues and Eigenvectors

- Covariance: the axes and the spread of an ellipse — M8, PCA in M4
- Loss curvature on a quadratic bowl: the largest eigenvalue caps the step, the ratio sets the rate — M6
- A linear iteration is stable, or it is not — next chapter, M7
- The principal moments of an inertia matrix
- Power iteration: the turning arrows as an algorithm, once one $|\lambda|$ is strictly largest



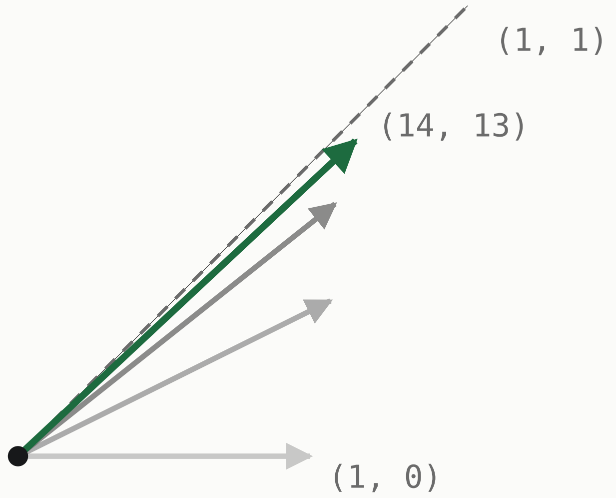
Eigenvalues and Eigenvectors

WORKED EXAMPLE

$x_0 = (1, 0)$: $Ax_0 = (2, 1)$, $A^2x_0 = (5, 4)$, $A^3x_0 = (14, 13)$.

Divide by the first entry: $(1, 0), (1, \frac{1}{2}), (1, \frac{4}{5}), (1, \frac{13}{14})$.

First entries grow by $2, \frac{5}{2}, \frac{14}{5}$ toward 3, the strictly largest $|\lambda|$. From $(1, -1)$ nothing turns.



Directions only; each arrow is drawn to its own scale.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eigenvalues and Eigenvectors

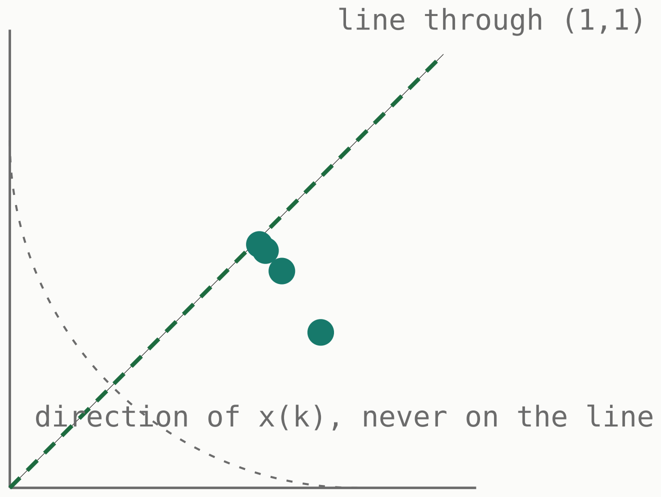
Two directions and two factors. Not yet: the hundredth state without a hundred products.

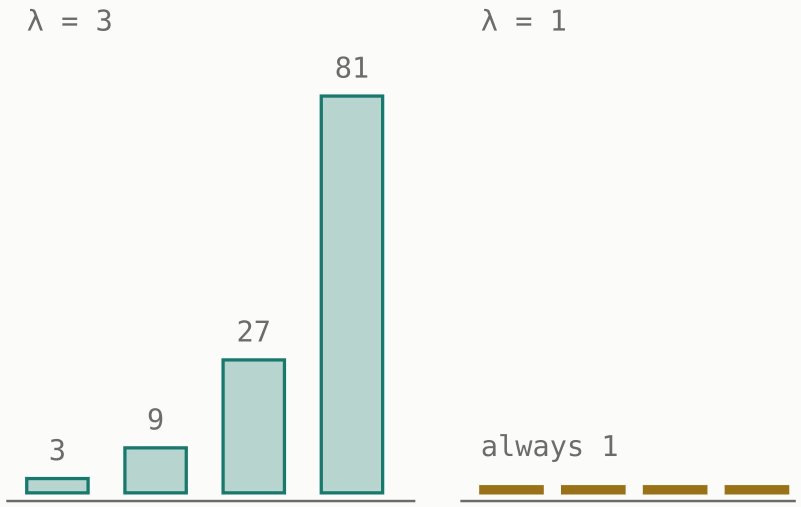
$$x_{k+1} = Ax_k \implies x_k = A^k x_0$$

x_k	the state after k steps
`build` A^k	one at a time, $k-1$ products, about n^3 each
`only` $A^k x_0$	k matrix–vector products, about n^2 each

x_0		x_1		x_2				x_k
(4, 2)	→	(10, 8)	→	(28, 26)	→	. . .	→	?

Diagonalization and Modes

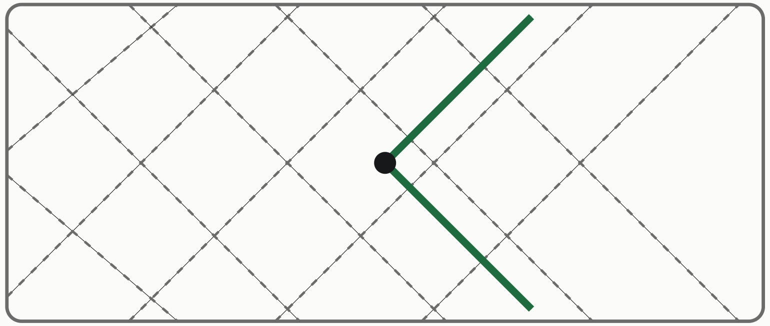




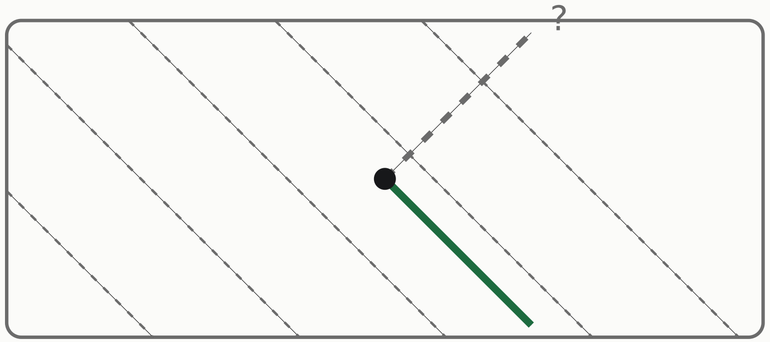
Diagonalization and Modes

DIAGONALIZABLE A is diagonalizable $\iff$ $\mathbb{R}^n$ has a basis of eigenvectors of A

a basis of eigenvectors	n independent directions, each one only stretched by A
the consequence	in that basis the matrix of the operator is diagonal



TWO DIRECTIONS: A BASIS



ONE DIRECTION: NOT A BASIS

Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.50 diagonalizable, p. 163; 5.55 conditions equivalent to diagonalizability, p. 165. Springer, 2024. Open access.

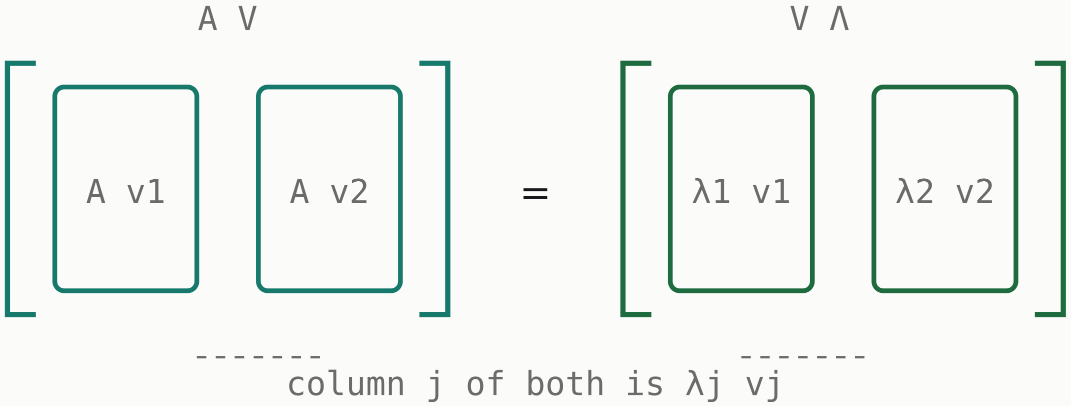
Diagonalization and Modes

$$AV = V\Lambda \implies A = V\Lambda V^{-1}$$

V the eigenvectors as columns, invertible because they are a basis

Λ the eigenvalues on the diagonal, in the same order as those columns

step two multiply both sides on the right by V^{-1}



Goodfellow, I., Bengio, Y. & Courville, A. ***Deep Learning***, ch. 2.7 eigendecomposition. MIT Press, 2016.

Diagonalization and Modes

WORKED EXAMPLE

$$V\Lambda = \begin{bmatrix}3 & 1\\ 3 & -1\end{bmatrix}, \qquad (V\Lambda)V^{-1} = \frac{1}{2}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = A$$

$V\Lambda$ column j of V multiplied by λ_j, which is 3 and 1

the half $V^{-1} = \frac{1}{2}V$ for this V, so it rides to the end

$$A = \underset{V}{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}} \underset{\Lambda,\ \text{DIAGONAL}}{\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}} \tfrac{1}{2} \underset{V^{-1}}{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}} \dashrightarrow \begin{matrix}2 & 1\\ 1 & 2\end{matrix}$$

Diagonalization and Modes

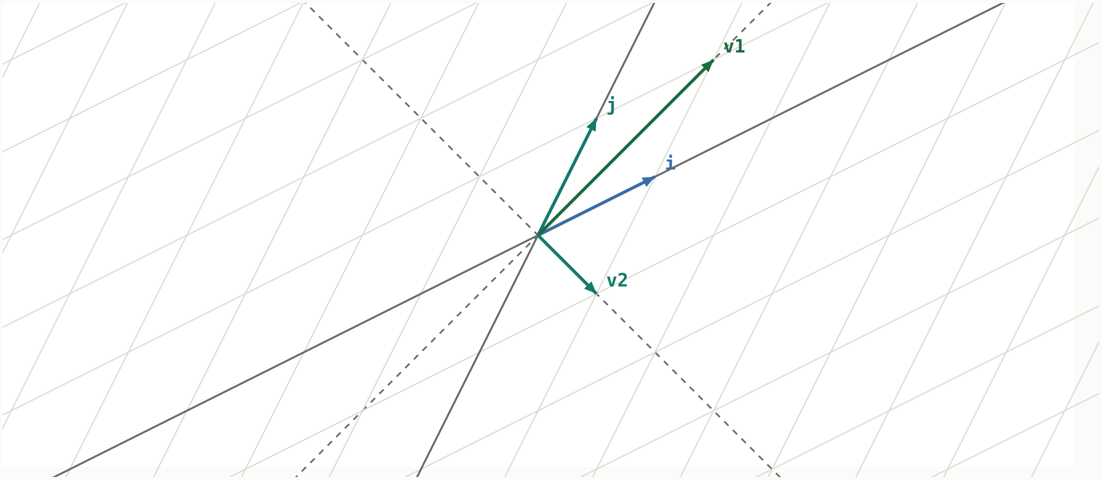
CHANGE OF BASIS $\Lambda = V^{-1}AV = [A]_{\mathcal{B}}$

$V^{-1}AV$ similarity, run with the eigenvectors as the new basis

$[A]_{\mathcal{B}}$ the matrix of the operator in the basis $\mathcal{B}$

Λ one operator, three names

$$\begin{tikzcd} {[x]_B} \arrow[r, dashed, "\Lambda"] & {[A\ x]_B} \arrow[d, dashed, "V"] \\ x \arrow[u, dashed, "V^{-1}"] \arrow[r, dashed, "A"'] & A\ x \end{tikzcd}$$



The dashed lines are the columns of V — the axes Λ is written in.

Diagonalization and Modes

$$A^k = V\Lambda^k V^{-1}, \qquad \Lambda^k = \mathrm{diag}(\lambda_1^k, \ldots, \lambda_n^k)$$

the middle $V^{-1}V = I$ cancels $k-1$ times

Λ^k raise each diagonal entry; nothing else moves

the cost after the factorization, all dependence on k is n scalar powers

$$A \cdot A \cdot A = V \quad \Lambda \quad \underbrace{\cancel{V^{-1} \quad V}}_{= I} \quad \Lambda \quad \underbrace{\cancel{V^{-1} \quad V}}_{= I} \quad \Lambda \quad V^{-1}$$

$= V \quad \Lambda^3 \quad V^{-1}$ `two factors survive, whatever k is`

Strang, G. ***18.06 Linear Algebra***, Lecture 22, diagonalization and powers of A. MIT OpenCourseWare, 2010.

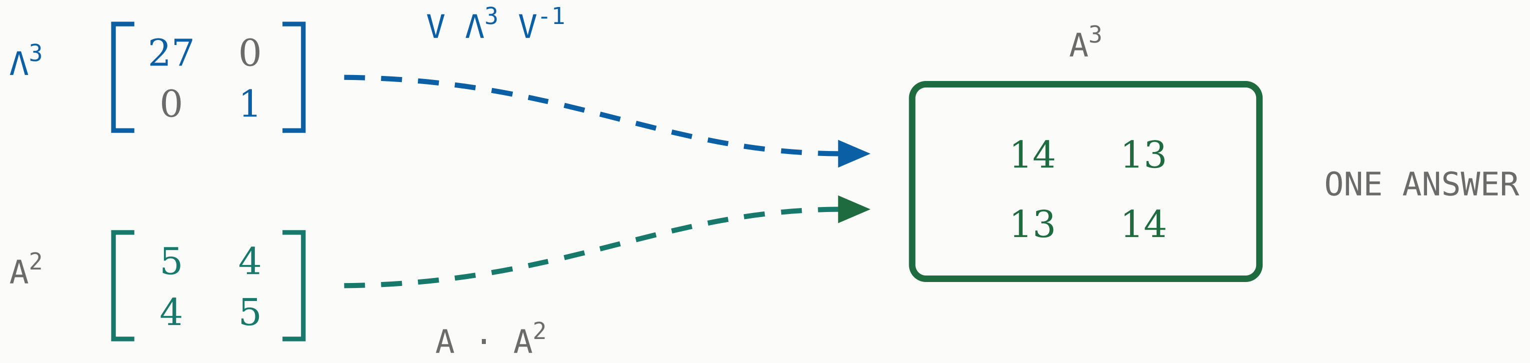
Diagonalization and Modes

WORKED EXAMPLE

$$A^k = V\Lambda^k V^{-1} = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}$$

$k = 3$ $3^3 = 27$, so $(27+1)/2 = 14$ and $(27-1)/2 = 13$

check square A, then multiply by A once more



$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Diagonalization and Modes

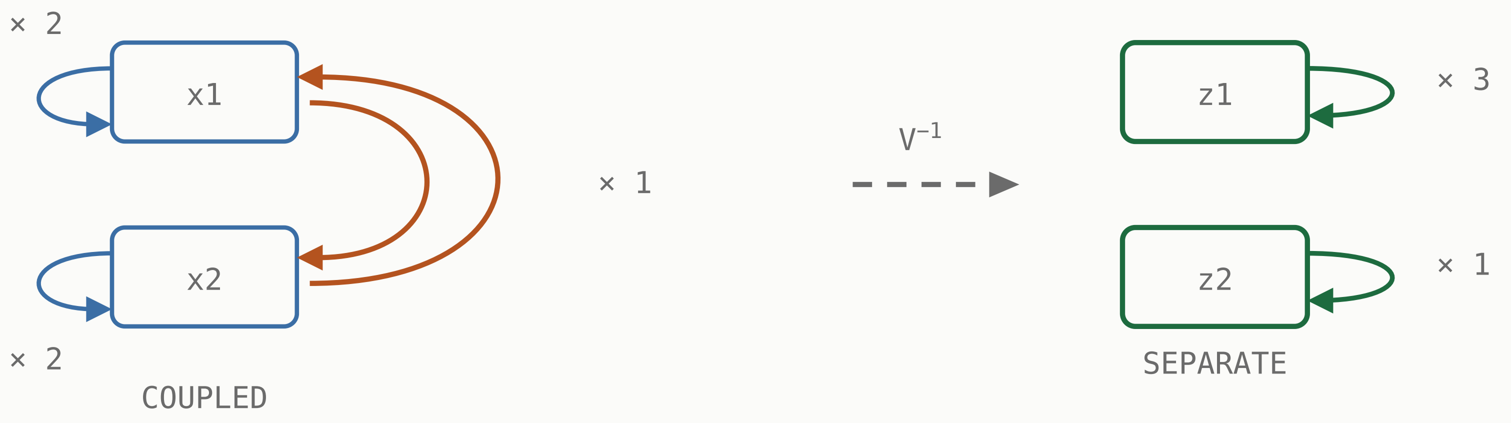
$$x_{k+1} = Ax_k, \qquad z_k = V^{-1}x_k$$

MODAL COORDINATES

$$z_{k+1} = V^{-1}Ax_k = (V^{-1}AV)\,z_k = \Lambda z_k \implies z_k = \Lambda^k z_0$$

z the state in eigen-coordinates, one number per mode

$\Lambda = \mathrm{diag}(3, 1)$ no coupling: row i mentions only z_i



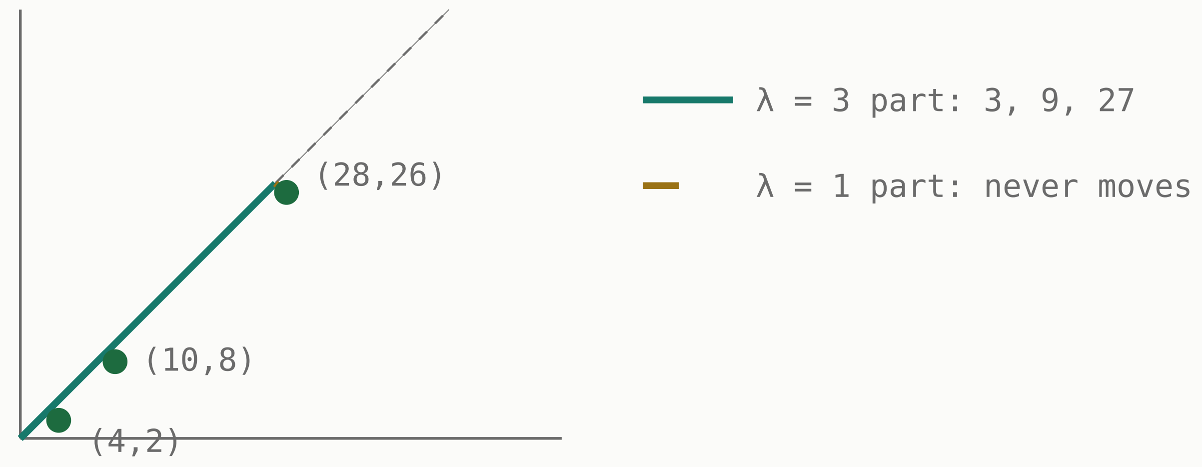
Strang, G. ***18.06 Linear Algebra***, Lecture 22, diagonalization and powers of A. MIT OpenCourseWare, 2010. CC BY-NC-SA 4.0.

Diagonalization and Modes

WORKED EXAMPLE

$$x_k = V\Lambda^k z_0 = 3 \cdot 3^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot 1^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

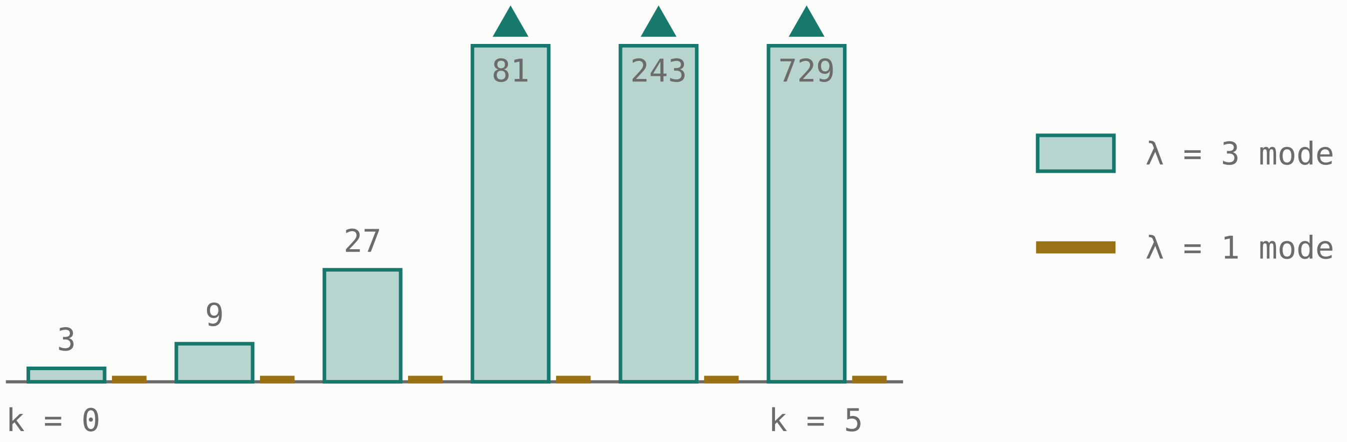
$z_0 = (3, 1)$	$V^{-1}x_0$, the coordinates already found for $x_0 = (4, 2)$
$k = 1, 2$	$9(1,1) + (1,-1) = (10,8)$, $27(1,1) + (1,-1) = (28,26)$
every k	the entries differ by 2: the dots sit on one straight line



Diagonalization and Modes

$$x_k = \sum_j c_j \lambda_j^k v_j, \qquad \lambda_1^{-k} x_k \longrightarrow c_1 v_1$$

λ_1	strictly largest in absolute value, and nonzero
$c_1 \neq 0$	x_0 must carry some v_1
residual	here $x_k - 3 \cdot 3^k(1,1) = (1,-1)$ at every k
at $k = 10$	$3 \cdot 3^{10} = 177{,}147$ against 1



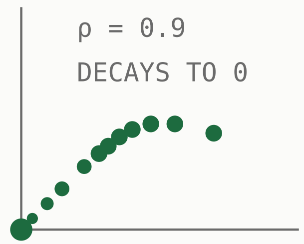
Diagonalization and Modes

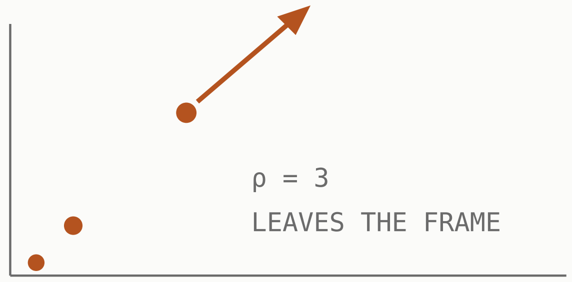
$$\rho(A) = \max_j |\lambda_j|; \qquad \text{diagonalizable } A: \; A^k x_0 \to 0 \;\; \forall x_0 \iff \rho(A) < 1$$

ρ	the largest of the $|\lambda_j|$, over every eigenvalue
x_0	any starting state
`strict`	$\rho = 1$ is not enough

WORKED EXAMPLE

A_d has 0.7 on the diagonal and 0.2 off it: the same eigenvectors, and eigenvalues 0.9 and 0.5.



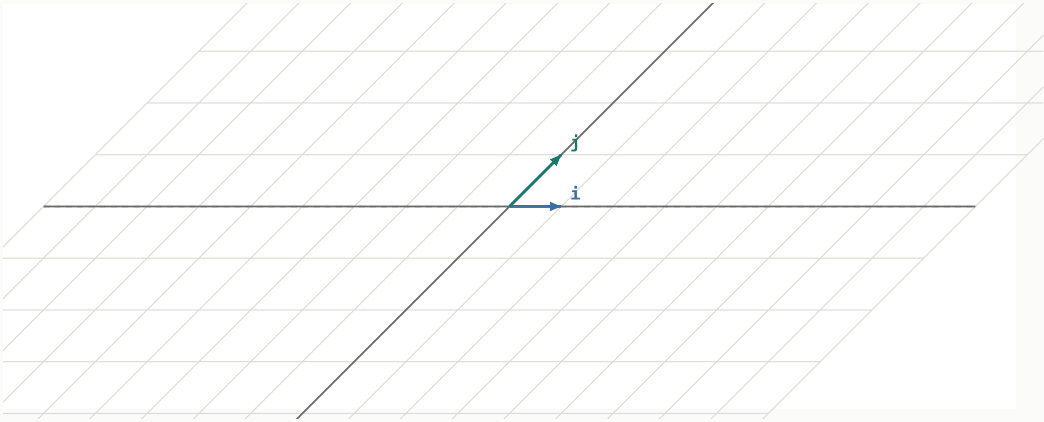


Strang, G. ***18.06 Linear Algebra***, Lecture 22, diagonalization and powers of A. MIT OpenCourseWare, 2010.

Diagonalization and Modes

DEFECTIVE $$(\mathrm{Sh} - I)v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = 0 \;\Rightarrow\; v_2 = 0 \;\Rightarrow\; v \in \mathrm{span}\{(1,0)\}$$

Sh	the shear in the corner: x slides by y
$\lambda = 1$ twice	from $(1-\lambda)^2$, yet one direction solves it
defective	short of a basis of eigenvectors, so not diagonalizable



One dashed invariant line, where two would have made a basis.

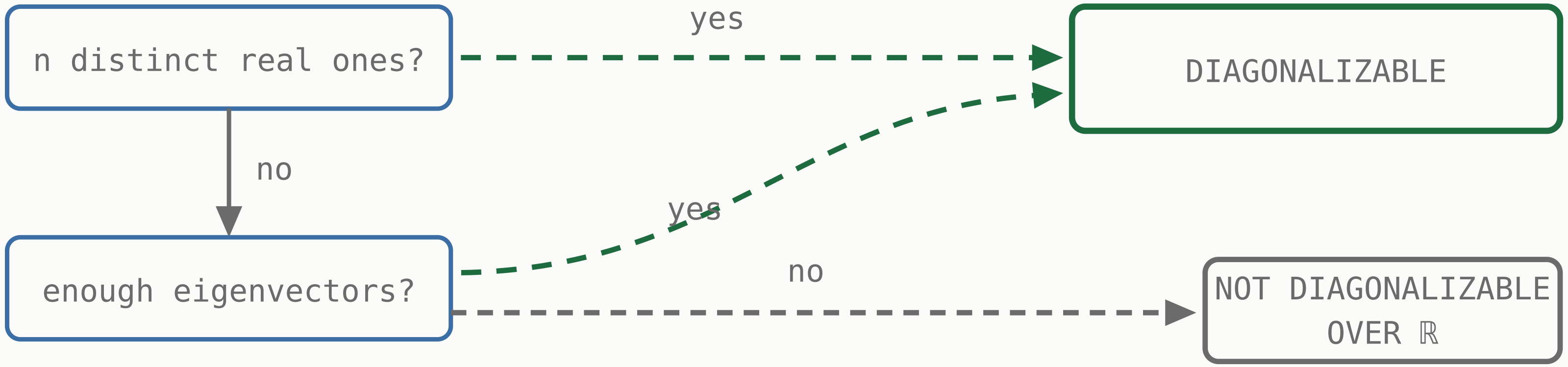
Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.55 conditions equivalent to diagonalizability, p. 165. Springer, 2024. Open access.

Diagonalization and Modes

$$n \text{ distinct real eigenvalues} \Rightarrow A \text{ is diagonalizable over } \mathbb{R}$$

`real`	slide 62's rotation: distinct complex roots, no real basis
`one way only`	the identity repeats an eigenvalue, yet is diagonal

Over $\mathbb{C}$ every square matrix has a Jordan form: diagonal plus ones just above it.

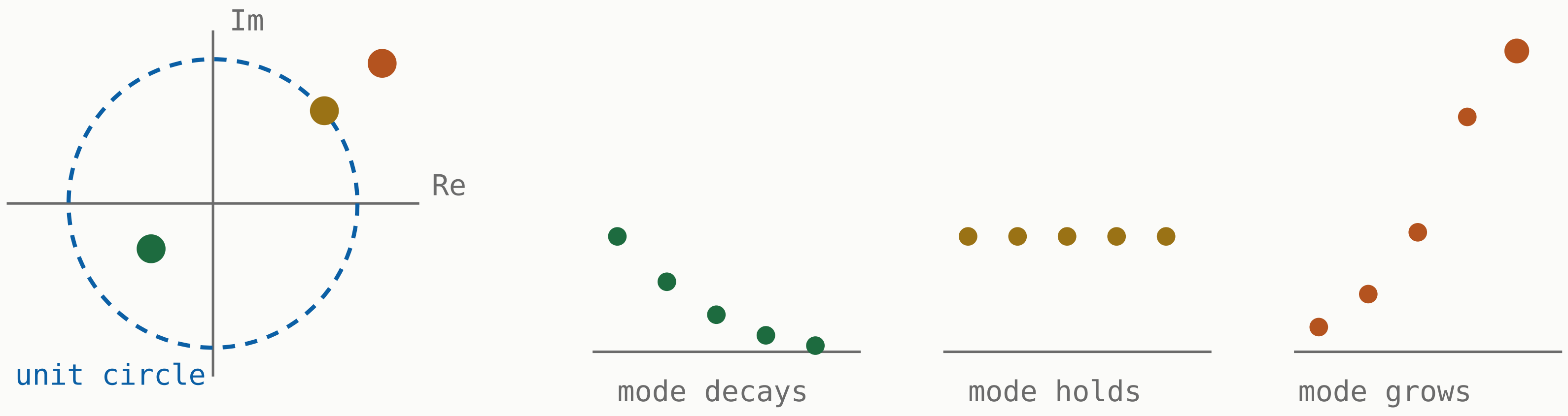


Axler, S. ***Linear Algebra Done Right***, 4th ed., 5.58, p. 166. Springer, 2024. Jordan form quoted without proof: Horn, R. & Johnson, C. ***Matrix Analysis***, 2nd ed. Cambridge, 2012.

Diagonalization and Modes

$$x_{k+1} = (A - BK)x_k \to 0 \ \forall x_0 \iff \rho(A - BK) < 1$$

A, B	the plant: how the state drifts, where the input enters
K	the gain you design; it moves the eigenvalues
$\rho < 1$	strictly inside; a repeated root on the circle can still grow
`next door`	M7 does $\mathrm{Re}\,\lambda < 0$; M8 pushes a covariance through A



RECAP

The five questions, answered

1 · COORDINATES

what are these numbers
written in?

2 · ROTATIONS

which change of basis
is a turn?

3 · DETERMINANT

does it collapse,
and does it flip?

4 · EIGENVECTORS

which directions
survive?

5 · DIAGONALIZATION

what is A to the k?

Courses and books

1. Axler, S. *Linear Algebra Done Right*, 4th ed. Springer, 2024. Open access. Quotes include 3.84, 5.5, 5.52, 9.50, 9.61.
2. Strang, G. *18.06 Linear Algebra*, L19, 20, 21, 22, 28, 31. MIT OCW, 2010. CC BY-NC-SA 4.0.
3. Lynch and Park. *Modern Robotics*. Cambridge, 2017.
4. Grizzle, J. *ROB 501*, notes 11. Michigan, 2018. CC BY-NC 4.0.
5. Sanderson, G. *Essence of Linear Algebra*, ch. 6. 3Blue1Brown, 2016.

Books carried forward

6. Goodfellow, Bengio and Courville. *Deep Learning*, ch. 2.7 and 2.8. MIT Press, 2016.
7. Horn and Johnson. *Matrix Analysis*, 2nd ed. Cambridge, 2012. Jordan form, slide 80.
8. Strang, G. *Linear Algebra and Learning from Data*. Wellesley–Cambridge, 2019. Reading for M3 and M4.
9. The M1 deck, for the null space and rank this meeting assumes.