EXERCISE · SLIDE 6 · QUESTION

Vector Fields

Your turn: a changed decay rate

$$\dot{x} = -3x, \qquad x(0) = 2$$

x	scalar state
−3	fixed decay rate
2	initial value

Which trajectory satisfies BOTH the rate and the start?

A. exp(−3t)

B. 2 exp(−3t)

C. 2 exp(3t)

Exponential decay — [4] FNC §6.1 · Course example

Vector Fields

Your turn: a changed decay rate

Solution 1 / 4

1 · Solve the rate equation

x(t): scalar state at time t · c: constant set by the initial value

1. For a constant decay rate −3, the exponential family is

$$x(t) = ce^{-3t}, \qquad \dot{x}(t) = -3ce^{-3t} = -3x(t).$$

Exponential decay — [4] FNC §6.1 · Course example

Vector Fields

Your turn: a changed decay rate

Solution 2 / 4

2 · Impose the initial value

x(t): scalar state at time t · c: constant set by the initial value

1. At time zero the exponential is one, so the constant must be two.

$$x(0) = ce^{0} = c = 2 \quad \Longrightarrow \quad x(t) = 2e^{-3t}.$$

Exponential decay — [4] FNC §6.1 · Course example

Vector Fields

Your turn: a changed decay rate

Solution 3 / 4

3 · Verify both conditions

x(t): scalar state at time t · c: constant set by the initial value

1. The chosen trajectory starts at two and has the required derivative.

$$x(0) = 2, \qquad \dot{x}(t) = -6e^{-3t} = -3x(t).$$

Exponential decay — [4] FNC §6.1 · Course example

EXERCISE · SLIDE 6 · SOLUTION 4/4

Vector Fields

Your turn: a changed decay rate

Solution 4 / 4

Answer · B

x(t): scalar state at time t · c: constant set by the initial value

---

1. A starts at one, not two.
2. B has the correct initial value and decay rate.
3. C reverses the sign of the rate.

Exponential decay — [4] FNC §6.1 · Course example

EXERCISE · SLIDE 6 · CHECK THE RESULT

Vector Fields

Your turn: a changed decay rate

$$\dot{x} = -3x, \qquad x(0) = 2$$

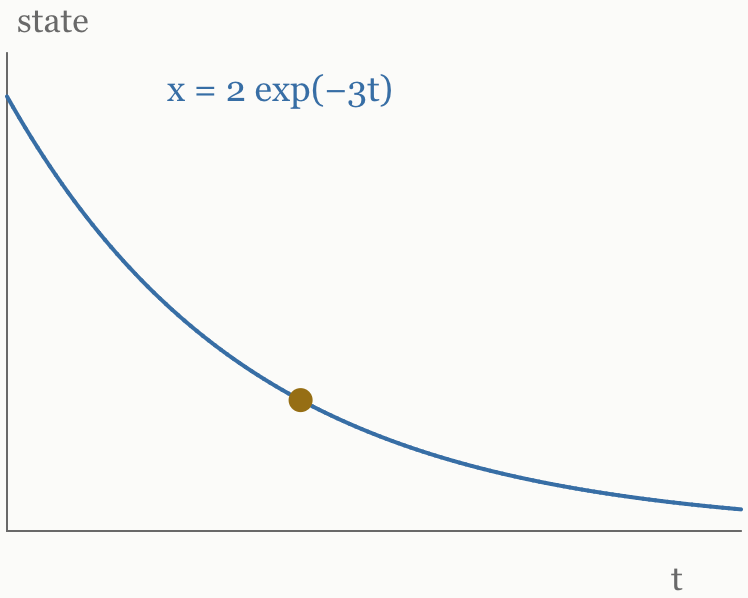
- x scalar state
- −3 fixed decay rate
- 2 initial value

Which trajectory satisfies BOTH the rate and the start?

A. exp(−3t)

B. 2 exp(−3t)

C. 2 exp(3t)



✓

Correct B. Check start AND rate.

Exponential decay — [4] FNC §6.1 · Course example

EXERCISE · SLIDE 11 · QUESTION

Robot State

Your turn: reduce the applied force

$$M = 2,\ C = .5,\ g = 3,\ \tau = 1,\ v = 2$$
$$Ma = \tau - Cv - g, \quad \dot{x} = (v, a)^T$$

`a`	acceleration
`x=(q,v)ᵀ`	position then velocity

Which state rate follows from the force balance?

A. $(-1.5, 2)^T$

B. $(2, -3)^T$

C. $(2, -1.5)^T$

Forward dynamics — [2] Modern Robotics §8.1 · Course example

Robot State

Your turn: reduce the applied force

Solution 1 / 5

1 · Compute the velocity force

q,v,a: position, velocity and acceleration · M,C: inertia and velocity-force coefficient · τ,g: applied force and gravity force

1. The given coefficient is 0.5 and the velocity is 2.

$$Cv = 0.5 \times 2 = 1.$$

Forward dynamics — [2] Modern Robotics §8.1 · Course example

EXERCISE · SLIDE 11 · SOLUTION 2/5

Robot State

Your turn: reduce the applied force

Solution 2 / 5

2 · Apply force balance

q,v,a: position, velocity and acceleration · M,C: inertia and velocity-force coefficient · τ,g: applied force and gravity force

1. Subtract the velocity force and gravity from the applied force.

$$Ma = \tau - Cv - g = 1 - 1 - 3 = -3.$$

Forward dynamics — [2] Modern Robotics §8.1 · Course example

Robot State

Your turn: reduce the applied force

Solution 3 / 5

3 · Divide by inertia

q,v,a: position, velocity and acceleration · M,C: inertia and velocity-force coefficient · τ,g: applied force and gravity force

1. Acceleration is net force divided by inertia; it is not the net force itself.

$$a = \frac{-3}{2} = -1.5.$$

Forward dynamics — [2] Modern Robotics §8.1 · Course example

Robot State

Your turn: reduce the applied force

Solution 4 / 5

4 · Keep the state order

q,v,a: position, velocity and acceleration · M,C: inertia and velocity-force coefficient · τ,g: applied force and gravity force

1. Position differentiates to velocity; velocity differentiates to acceleration.

$$x = (q, v)^T \quad \Longrightarrow \quad \dot{x} = (v, a)^T = (2, -1.5)^T.$$

Forward dynamics — [2] Modern Robotics §8.1 · Course example

EXERCISE · SLIDE 11 · SOLUTION 5/5

Robot State

Your turn: reduce the applied force

Solution 5 / 5

Answer · C

q,v,a: position, velocity and acceleration · M,C: inertia and velocity-force coefficient · τ,g: applied force and gravity force

1. A swaps velocity and acceleration.
2. B omits division by inertia two.
3. C preserves component order and divides net force by inertia.

Forward dynamics — [2] Modern Robotics §8.1 · Course example

EXERCISE · SLIDE 11 · CHECK THE RESULT

Robot State

Your turn: reduce the applied force

$$M = 2,\ C = .5,\ g = 3,\ \tau = 1,\ v = 2$$
$$Ma = \tau - Cv - g, \quad \dot{x} = (v, a)^T$$

`a` acceleration

`x=(q,v)ᵀ` position then velocity

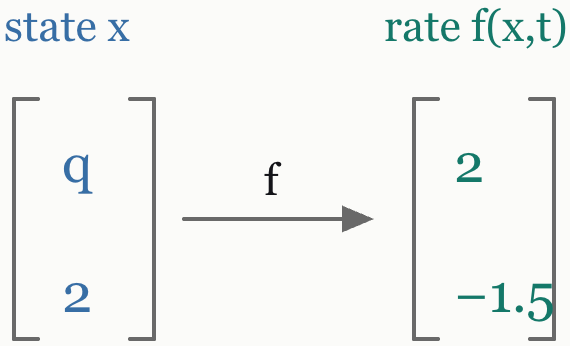
Which state rate follows from the force balance?

A. $(-1.5,2)^T$

B. $(2,-3)^T$

C. $(2,-1.5)^T$

> ✓
>
> Correct C. Velocity, then acceleration.



position rate = velocity

velocity rate = acceleration

Forward dynamics — [2] Modern Robotics §8.1 · Course example

EXERCISE · SLIDE 16 · QUESTION

Linear Dynamics

Return from modal coordinates

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Lambda = \mathrm{diag}(-1, -2)$$

- V eigenvector basis
- Λ matching eigenvalues
- c_0 coordinates of x_0 in the V basis

For NEW $x_0=(1,1)^T$, which modal description is correct?

A. $c_0=(0,1)^T$; only the -2 mode

B. $c_0=(1,2)^T$; both modes

C. $c_0=(1,1)^T$; both modes

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

Linear Dynamics

Return from modal coordinates

Solution 1 / 5

1 · Invert the coordinate map

x_0: new initial state $(1,1)^T$ · V,Λ: eigenvector matrix and matching eigenvalues · $c_0,c(t)$: initial and evolving modal coordinates

1. Because $x=Vc$, solve for c using the inverse, not the transpose.

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

Linear Dynamics

Return from modal coordinates

Solution 2 / 5

2 · Transform the new start

x_0: new initial state $(1,1)^T$ · V,Λ: eigenvector matrix and matching eigenvalues · $c_0,c(t)$: initial and evolving modal coordinates

1. Multiply each row of the inverse by the initial column.

$$c_0 = V^{-1}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1-1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

Linear Dynamics

Return from modal coordinates

Solution 3 / 5

3 · Evolve the active mode

x_0: new initial state $(1,1)^T$ · V,Λ: eigenvector matrix and matching eigenvalues · $c_0,c(t)$: initial and evolving modal coordinates

1. The coefficient of the -1 mode is zero; only the -2 mode remains.

$$c(t) = \begin{bmatrix} 0e^{-t} \\ 1e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}.$$

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

Linear Dynamics

Return from modal coordinates

Solution 4 / 5

4 · Return to state coordinates

x_0: new initial state $(1,1)^T$ · V,Λ: eigenvector matrix and matching eigenvalues · $c_0,c(t)$: initial and evolving modal coordinates

1. Both state components are equal even though only one modal coordinate is active.

$$x(t) = Vc(t) = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

EXERCISE · SLIDE 16 · SOLUTION 5/5

Linear Dynamics

Return from modal coordinates

Solution 5 / 5

Answer · A

x_0: new initial state $(1,1)^T$ · V,Λ: eigenvector matrix and matching eigenvalues · $c_0,c(t)$: initial and evolving modal coordinates

1. A recognizes the second eigenvector.
2. B uses the transpose instead of the inverse of a nonorthogonal basis.
3. C copies state coordinates without changing basis.

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

EXERCISE · SLIDE 16 · CHECK THE RESULT

Linear Dynamics

Return from modal coordinates

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Lambda = \operatorname{diag}(-1, -2)$$

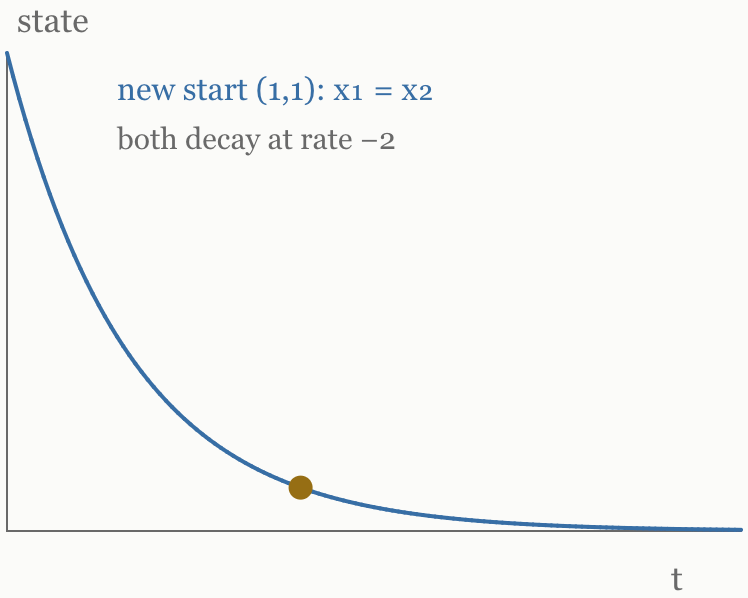
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- Λ matching eigenvalues
- c_0 coordinates of x_0 in the V basis

For NEW $x_0=(1,1)^T$, which modal description is correct?

A. $c_0=(0,1)^T$; only the -2 mode

B. $c_0=(1,2)^T$; both modes

C. $c_0=(1,1)^T$; both modes



✓

Correct A. Use the inverse basis.

Change of basis; matrix exponential — [3] Strang, lecture 23 · Course derivation

EXERCISE · SLIDE 18 · QUESTION

Linear Dynamics

A defective system

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad x_0 = (0,1)^T, \quad x(t) = (te^{-t}, e^{-t})^T$$

`defective` fewer than n independent eigenvectors in $\mathbb{C}^n$

`v` eigenvector; here v lies along $(1,0)^T$

Does x(t) satisfy ẋ=Ax and x(0)=x0?

A. No: a defective matrix has no solution.

B. No: the factor t forces growth forever.

C. Yes: its derivative and initial state both match.

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

Linear Dynamics

A defective system

Solution 1 / 5

1 · Differentiate the candidate

A: given constant state matrix · x(t),x0: candidate trajectory and required initial state · t: continuous time

1. Use the product rule for the first component.

$$\frac{d}{dt}(te^{-t}) = e^{-t} - te^{-t} = (1-t)e^{-t}, \qquad \frac{d}{dt}e^{-t} = -e^{-t}.$$

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

Linear Dynamics

A defective system

Solution 2 / 5

2 · Compute the model rate

A: given constant state matrix · x(t),x0: candidate trajectory and required initial state · t: continuous time

1. Multiply A by the candidate and compare component by component.

$$Ax = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} (1-t)e^{-t} \\ -e^{-t} \end{bmatrix} = \dot{x}.$$

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

Linear Dynamics

A defective system

Solution 3 / 5

3 · Check the start

A: given constant state matrix · x(t),x0: candidate trajectory and required initial state · t: continuous time

1. The candidate satisfies the initial value as well as the differential equation.

$$x(0) = \begin{bmatrix} 0e^{0} \\ e^{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_0.$$

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

Linear Dynamics

A defective system

Solution 4 / 5

4 · Interpret the solution

A: given constant state matrix · x(t),x0: candidate trajectory and required initial state · t: continuous time · v: eigenvector · I: identity matrix

1. Here λ=−1 is repeated. Its eigenvectors have only one independent direction.

$$(A+I)v=0 \;\Rightarrow\; v_2=0 \;\Rightarrow\; v=c(1,0)^T, \quad c\neq 0.$$

2. The exponential eventually dominates t. The matrix exponential gives the solution even without an eigenbasis.

$$\lim_{t\to\infty} te^{-t}=0, \qquad x(t)=e^{At}x_0.$$

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

EXERCISE · SLIDE 18 · SOLUTION 5/5

Linear Dynamics

A defective system

Solution 5 / 5

Answer · C

A: given constant state matrix · x(t),x0: candidate trajectory and required initial state · t: continuous time

1. A confuses an incomplete eigenbasis with absence of a solution.
2. B ignores the exponential decay multiplying t.
3. C checks both defining conditions.

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

EXERCISE · SLIDE 18 · CHECK THE RESULT

Linear Dynamics

A defective system

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad x_0 = (0,1)^T, \quad x(t) = (te^{-t}, e^{-t})^T$$

`defective` fewer than n independent eigenvectors in $\mathbb{C}^n$

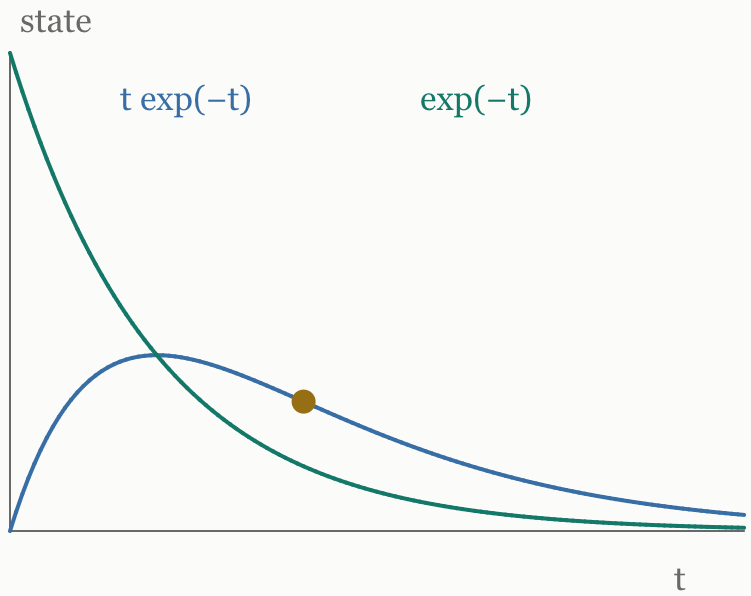
`v` eigenvector; here v lies along $(1,0)^T$

Does x(t) satisfy ẋ=Ax and x(0)=x0?

A. No: a defective matrix has no solution.

B. No: the factor t forces growth forever.

C. Yes: its derivative and initial state both match.



✓

Correct C. Both conditions hold.

Defective matrix; matrix exponential — [3] Strang, lecture 23 · [7] NIST DLMF §1.2 · Course example

EXERCISE · SLIDE 20 · QUESTION

Linear Dynamics

Zero real parts need a closer look

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad x_0 = (0, 1)^T$$

`A`	both eigenvalues zero
`x₀`	chosen initial state
`x(t)`	state to determine

What does this zero-eigenvalue example do?

A. $x(t)=(0,1)^T$: fixed

B. $x(t)=(t,1)^T$: unbounded

C. $x(t)=(\exp(t),1)^T$: exponential growth

Boundedness; asymptotic decay — [3] Strang, lecture 23

EXERCISE · SLIDE 20 · SOLUTION 1/5

Linear Dynamics

Zero real parts need a closer look

Solution 1 / 5

1 · Read the component equations

$x_1(t), x_2(t)$: first and second coordinates; subscripts here label coordinates · A: given matrix coupling coordinate 2 into coordinate 1 · s: integration variable

1. Matrix multiplication gives one constant coordinate and one driven coordinate.

$$\dot{x} = Ax \quad \Longrightarrow \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = 0.$$

Boundedness; asymptotic decay — [3] Strang, lecture 23

Linear Dynamics

Zero real parts need a closer look

Solution 2 / 5

2 · Solve the second coordinate

$x_1(t), x_2(t)$: first and second coordinates; subscripts here label coordinates · A: given matrix coupling coordinate 2 into coordinate 1 · s: integration variable

1. Zero derivative means constant; its initial value is one.

$$x_2(t) = x_2(0) + \int_0^t 0 \, ds = 1.$$

Boundedness; asymptotic decay — [3] Strang, lecture 23

Linear Dynamics

Zero real parts need a closer look

Solution 3 / 5

3 · Integrate the first coordinate

$x_1(t), x_2(t)$: first and second coordinates; subscripts here label coordinates · A: given matrix coupling coordinate 2 into coordinate 1 · s: integration variable

1. The first coordinate starts at zero and has constant derivative one.

$$x_1(t) = x_1(0) + \int_0^t 1\, ds = 0 + t = t.$$

Boundedness; asymptotic decay — [3] Strang, lecture 23

Linear Dynamics

Zero real parts need a closer look

Solution 4 / 5

4 · Classify the result

$x_1(t), x_2(t)$: first and second coordinates; subscripts here label coordinates · A: given matrix coupling coordinate 2 into coordinate 1 · s: integration variable

1. The state grows linearly, despite both eigenvalues being zero.

$$x(t) = (t, 1)^T, \qquad \|x(t)\|_2 = \sqrt{t^2 + 1} \to \infty.$$

Boundedness; asymptotic decay — [3] Strang, lecture 23

EXERCISE · SLIDE 20 · SOLUTION 5/5

Linear Dynamics

Zero real parts need a closer look

Solution 5 / 5

Answer · B

$x_1(t), x_2(t)$: first and second coordinates; subscripts here label coordinates · A: given matrix coupling coordinate 2 into coordinate 1 · s: integration variable

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1. A ignores coupling between coordinates.
2. B integrates the constant second coordinate into linear growth.
3. C invents an exponential where growth is linear.

Boundedness; asymptotic decay — [3] Strang, lecture 23

EXERCISE · SLIDE 20 · CHECK THE RESULT

Linear Dynamics

Zero real parts need a closer look

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad x_0 = (0, 1)^T$$

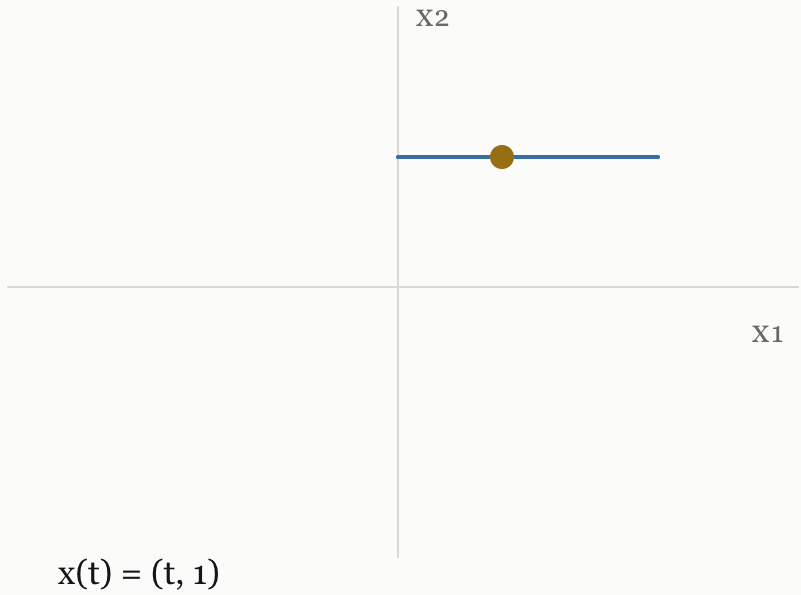
- `A` both eigenvalues zero
- `x0` chosen initial state
- `x(t)` state to determine

What does this zero-eigenvalue example do?

A. x(t)=(0,1)ᵀ: fixed

B. x(t)=(t,1)ᵀ: unbounded

C. x(t)=(exp(t),1)ᵀ: exponential growth



✓

Correct B. Coupling gives linear growth.

Boundedness; asymptotic decay — [3] Strang, lecture 23

EXERCISE · SLIDE 24 · QUESTION

Euler's Method

Your turn: two larger steps

$$\dot{x} = -2x, \quad x_0 = 1, \quad h = 0.2$$

- h fixed time step
- x_0 initial numerical state
- x_2 approximation at t=0.4

Which Euler pair uses a fresh rate at each step?

A. x1=.6, x2=.36

B. x1=.6, x2=.2

C. x1=.8, x2=.64

Explicit Euler method — [4] FNC §6.2 · Course example

EXERCISE · SLIDE 24 · SOLUTION 1/4

Euler's Method

Your turn: two larger steps

Solution 1 / 4

1 · Write the Euler update

x_k: accepted numerical state after k steps · h, t_k: step size 0.2 and time $t_k = kh$ · f(x): state rate $-2x$

1. Each step uses the rate at the current accepted state.

$$x_{k+1} = x_k + hf(x_k) = x_k + 0.2(-2x_k) = 0.6x_k.$$

Explicit Euler method — [4] FNC §6.2 · Course example

Euler's Method

Your turn: two larger steps

Solution 2 / 4

2 · Take the first step

x_k: accepted numerical state after k steps · h, t_k: step size 0.2 and time $t_k = kh$ · f(x): state rate $-2x$

1. At $x_0 = 1$, the rate is -2.

$$f(x_0) = -2, \qquad x_1 = 1 + 0.2(-2) = 0.6.$$

Explicit Euler method — [4] FNC §6.2 · Course example

Euler's Method

Your turn: two larger steps

Solution 3 / 4

3 · Recompute the second rate

x_k: accepted numerical state after k steps · h, t_k: step size 0.2 and time $t_k = kh$ · f(x): state rate $-2x$

1. The current state is now 0.6, so the rate changes to -1.2.

$$f(x_1) = -2(0.6) = -1.2, \qquad x_2 = 0.6 + 0.2(-1.2) = 0.36.$$

Explicit Euler method — [4] FNC §6.2 · Course example

Euler's Method

Your turn: two larger steps

Solution 4 / 4

Answer · A

x_k: accepted numerical state after k steps · h, t_k: step size 0.2 and time $t_k = kh$ · $f(x)$: state rate $-2x$

1. A reevaluates the rate at the new state.
2. B freezes the initial rate for both steps.
3. C uses half the given step size.

Explicit Euler method — [4] FNC §6.2 · Course example

EXERCISE · SLIDE 24 · CHECK THE RESULT

Euler's Method

Your turn: two larger steps

$$\dot{x} = -2x, \quad x_0 = 1, \quad h = 0.2$$

- h fixed time step
- x_0 initial numerical state
- x_2 approximation at t=0.4

Which Euler pair uses a fresh rate at each step?

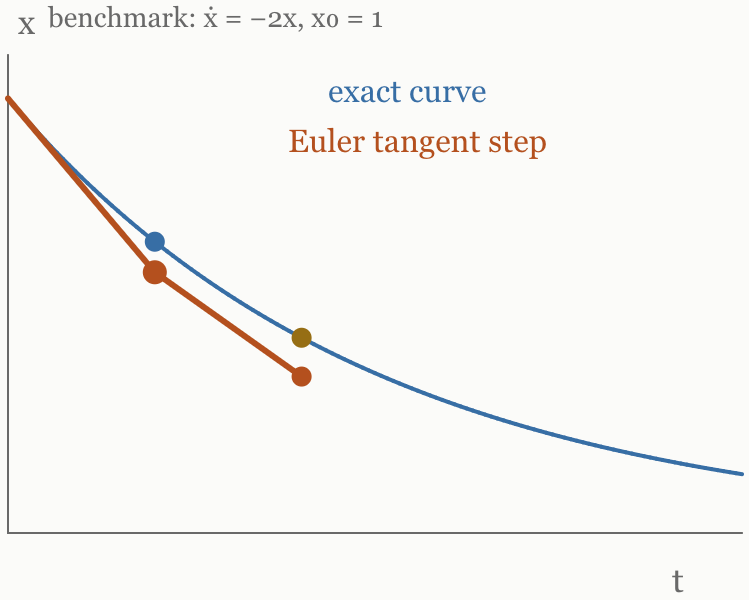
A. x1=.6, x2=.36

B. x1=.6, x2=.2

C. x1=.8, x2=.64

✓

Correct A. Reevaluate the rate.



Explicit Euler method — [4] FNC §6.2 · Course example

EXERCISE · SLIDE 28 · QUESTION

Numerical Stability

Predict three step-size regimes

$$\dot{x} = -4x, \quad h \in \{.25, .4, .6\}$$

- λ fixed decay rate
- h candidate steps
- x_0 any nonzero start

For h=.25, .4, .6, which Euler behaviors occur in order?

A. Zero; alternating growth; alternating growth

B. All decay because the ODE is stable

C. Zero; alternating decay; alternating growth

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 28 · SOLUTION 1/5

Numerical Stability

Predict three step-size regimes

Solution 1 / 5

1 · Derive the multiplier

h: positive time step · a: Euler multiplier · x_k: accepted numerical state; $x_0 \neq 0$

1. Its sign controls sign changes; its magnitude controls decay or growth.

$$x_{k+1} = x_k + h(-4x_k) = ax_k, \quad a = 1 - 4h, \quad x_k = a^k x_0.$$

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

Numerical Stability

Predict three step-size regimes

Solution 2 / 5

2 · Test h = 0.25

h: positive time step · a: Euler multiplier · x_k: accepted numerical state; $x_0 \neq 0$

1. A zero multiplier sends the state to zero in one step.

$$a = 1 - 4(0.25) = 0 \quad \Longrightarrow \quad x_1 = 0.$$

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

Numerical Stability

Predict three step-size regimes

Solution 3 / 5

3 · Test h = 0.4

h: positive time step · a: Euler multiplier · x_k: accepted numerical state; $x_0 \neq 0$

1. The negative sign alternates; the magnitude below one makes the states decay.

$$a = 1 - 4(0.4) = -0.6, \qquad |a| = 0.6 < 1.$$

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

Numerical Stability

Predict three step-size regimes

Solution 4 / 5

4 · Test h = 0.6

h: positive time step · a: Euler multiplier · x_k: accepted numerical state; $x_0 \neq 0$

1. The negative sign still alternates, but the magnitude above one now causes growth.

$$a = 1 - 4(0.6) = -1.4, \qquad |a| = 1.4 > 1.$$

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

Numerical Stability

Predict three step-size regimes

Solution 5 / 5

Answer · C

h: positive time step · a: Euler multiplier · x_k: accepted numerical state; $x_0 \neq 0$

1. A confuses a negative multiplier with growth.
2. B assumes physical stability guarantees numerical stability.
3. C checks the three multiplier magnitudes: zero, point six and one point four.

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 28 · CHECK THE RESULT

Numerical Stability

Predict three step-size regimes

$$\dot{x} = -4x, \quad h \in \{.25, .4, .6\}$$

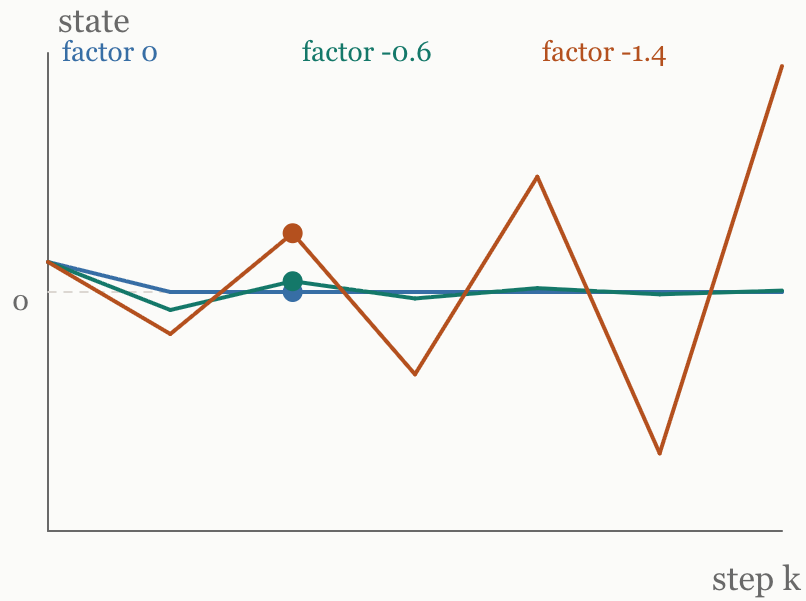
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A. Zero; alternating growth; alternating growth

B. All decay because the ODE is stable

C. Zero; alternating decay; alternating growth



✓

Correct C. Magnitude decides decay.

Step-size restriction; asymptotic decay — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 29 · QUESTION

Numerical Stability

Every mode must satisfy the bound

$$\dot{x} = Ax, \quad A = \mathrm{diag}(-2, -8)$$

- A constant state matrix
- h one step used for both modes
- x_0 arbitrary initial state

For Euler, which interval gives decay from EVERY start?

A. $0<h\le 1/4$

B. $0<h<1/4$

C. $0<h<1$

Stability of a linear system — [4] FNC §11.3 · Course example

Numerical Stability

Every mode must satisfy the bound

Solution 1 / 5

1 · Require both modes to decay

λ_1, λ_2: modal rates -2 and -8 · h: shared positive step size · a_i: Euler multiplier $1+h\lambda_i$ for mode i

1. An arbitrary initial state can contain either mode, so both multiplier magnitudes must be below one.

$$|1 + h\lambda_i| < 1 \quad \Longleftrightarrow \quad -1 < 1 + h\lambda_i < 1.$$

Stability of a linear system — [4] FNC §11.3 · Course example

Numerical Stability

Every mode must satisfy the bound

Solution 2 / 5

2 · Bound the slow mode

λ_1, λ_2: modal rates -2 and -8 · h: shared positive step size · a_i: Euler multiplier $1+h\lambda_i$ for mode i

1. Subtract one, divide by -2, and reverse both inequalities.

$$-1 < 1 - 2h < 1 \implies -2 < -2h < 0 \implies 0 < h < 1.$$

Stability of a linear system — [4] FNC §11.3 · Course example

Numerical Stability

Every mode must satisfy the bound

Solution 3 / 5

3 · Bound the fast mode

λ_1, λ_2: modal rates -2 and -8 · h: shared positive step size · a_i: Euler multiplier $1+h\lambda_i$ for mode i

1. The same calculation with -8 gives the more restrictive bound.

$$-1 < 1 - 8h < 1 \implies -2 < -8h < 0 \implies 0 < h < \tfrac{1}{4}.$$

Stability of a linear system — [4] FNC §11.3 · Course example

Numerical Stability

Every mode must satisfy the bound

Solution 4 / 5

4 · Intersect and check the endpoint

λ_1, λ_2: modal rates -2 and -8 · h: shared positive step size · a_i: Euler multiplier $1+h\lambda_i$ for mode i

1. The endpoint is excluded: its fast mode alternates with unchanged magnitude.

$$(0,1) \cap (0, \tfrac{1}{4}) = (0, \tfrac{1}{4}), \qquad h = \tfrac{1}{4} \Rightarrow a_2 = -1.$$

Stability of a linear system — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 29 · SOLUTION 5/5

Numerical Stability

Every mode must satisfy the bound

Solution 5 / 5

Answer · B

λ_1, λ_2: modal rates -2 and -8 · h: shared positive step size · a_i: Euler multiplier $1+h\lambda_i$ for mode i

1. A includes a multiplier minus one, so decay fails.
2. B intersects both strict modal bounds.
3. C ignores the restrictive fast mode.

Stability of a linear system — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 29 · CHECK THE RESULT

Numerical Stability

Every mode must satisfy the bound

$$\dot{x} = Ax, \quad A = \mathrm{diag}(-2, -8)$$

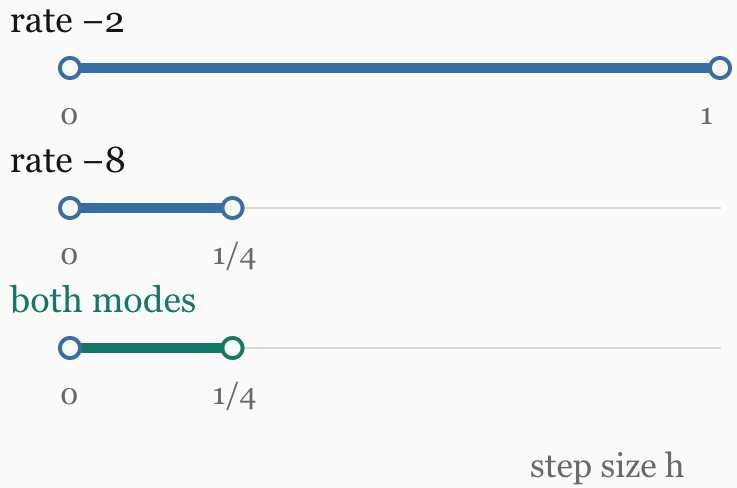
- `A` constant state matrix
- `h` one step used for both modes
- x_0 arbitrary initial state

For Euler, which interval gives decay from EVERY start?

A. $0<h\le 1/4$

B. $0<h<1/4$

C. $0<h<1$



✓

Correct B. Both strict bounds must hold.

Stability of a linear system — [4] FNC §11.3 · Course example

EXERCISE · SLIDE 35 · QUESTION

Runge–Kutta Methods

Your turn: halve the midpoint step

$$\dot{x} = -2x, \quad x(0) = x_0 = 1, \quad h = 0.1$$

`xtrial`	$x_0+(h/2)k_1$; temporary midpoint state
`k1,k2`	$f(x_0)$, $f(x_{trial})$: rates
`h k2`	estimated change over the whole step

Explicit midpoint: which x_1 estimates $x(0+h)$?

A. .9: xtrial

B. .72: xtrial + h k2

C. .82: x0 + h k2

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

EXERCISE · SLIDE 35 · SOLUTION 1/5

Runge–Kutta Methods

Your turn: halve the midpoint step

Solution 1 / 5

1 · Evaluate the initial rate

k1,k2: rates · xtrial: temporary state at t=0.05 · h k2: estimated full-step change · x1: next state at t=0.1

1. Use the original state x0=1.

$$k_1 = f(x_0) = -2(1) = -2.$$

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

Runge–Kutta Methods

Your turn: halve the midpoint step

Solution 2 / 5

2 · Build the trial midpoint

k1,k2: rates · xtrial: temporary state at t=0.05 · h k2: estimated full-step change · x1: next state at t=0.1

1. Advance only half a step using the initial rate.

$$x_{\text{trial}} = x_0 + \frac{h}{2} k_1 = 1 + 0.05(-2) = 0.9.$$

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

Runge–Kutta Methods

Your turn: halve the midpoint step

Solution 3 / 5

3 · Evaluate the midpoint rate

k1,k2: rates · xtrial: temporary state at t=0.05 · h k2: estimated full-step change · x1: next state at t=0.1

1. The second model call uses the trial state.

$$k_2 = f(x_{\text{trial}}) = -2(0.9) = -1.8.$$

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

Runge–Kutta Methods

Your turn: halve the midpoint step

Solution 4 / 5

4 · Compute the state at t = 0.1

k1,k2: rates · xtrial: temporary state at t=0.05 · h k2: estimated full-step change · x1: next state at t=0.1

1. The full-step change is $hk_2 = 0.1(-1.8) = -0.18$. Add it to the original state.

$$x_1 = x_0 + hk_2 = 1 - 0.18 = 0.82.$$

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

EXERCISE · SLIDE 35 · SOLUTION 5/5

Runge–Kutta Methods

Your turn: halve the midpoint step

Solution 5 / 5

Answer · C

k_1,k_2: rates · xtrial: temporary state at t=0.05 · h k_2: estimated full-step change · x_1: next state at t=0.1

1. A stops at the temporary trial state.
2. B adds the final increment to the wrong base state.
3. C uses the midpoint rate but updates from the original state.

Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

EXERCISE · SLIDE 35 · CHECK THE RESULT

Runge–Kutta Methods

Your turn: halve the midpoint step

$$\dot{x} = -2x, \quad x(0) = x_0 = 1, \quad h = 0.1$$

`xtrial`	$x_0+(h/2)k_1$; temporary midpoint state
`k1,k2`	$f(x_0)$, $f(x_{trial})$: rates
`h k2`	estimated change over the whole step

Explicit midpoint: which x_1 estimates $x(0+h)$?

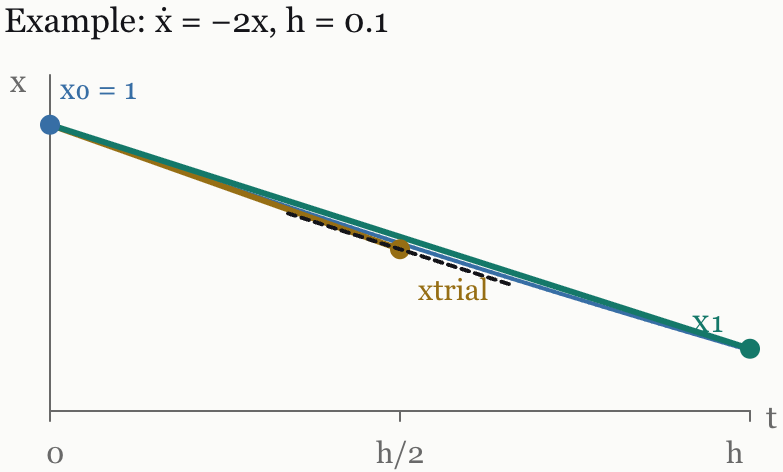
A. .9: xtrial

B. .72: xtrial + h k2

C. .82: x0 + h k2

✓

Correct C. Return to the original state.



Explicit midpoint method (RK2) — [4] FNC §6.4 · Course diagram

EXERCISE · SLIDE 38 · QUESTION

Runge–Kutta Methods

Your turn: compare equal model-call budgets

$$\dot{x} = -2x, \quad x_0 = 1, \quad h = 0.1$$

`h` one RK4 step interval

`cost` number of calls to f

How many Euler substeps match ONE RK4 step in model calls?

A. Four, each h/4, over the same interval

B. One, with step h

C. Four, each h, over the same interval

Function evaluations; order of accuracy — [4] FNC §6.4

EXERCISE · SLIDE 38 · SOLUTION 1/4

Runge–Kutta Methods

Your turn: compare equal model-call budgets

Solution 1 / 4

1 · Match the call budget

h: physical interval of one RK4 step; h=0.1 · nE,hE: Euler substep count and substep size · model call: one evaluation of f(x)=−2x · x4,E: state after four Euler substeps

1. Classical RK4 needs four model calls per step; Euler needs one.

$$n_E \times 1 = 4 \quad \Longrightarrow \quad n_E = 4.$$

Function evaluations; order of accuracy — [4] FNC §6.4

Runge–Kutta Methods

Your turn: compare equal model-call budgets

Solution 2 / 4

2 · Match the physical interval

h: physical interval of one RK4 step; h=0.1 · nE,hE: Euler substep count and substep size · model call: one evaluation of f(x)=−2x · x4,E: state after four Euler substeps

1. The four Euler substeps must add up to the same total time h.

$$4h_E = h \quad \Longrightarrow \quad h_E = h/4 = 0.025.$$

Function evaluations; order of accuracy — [4] FNC §6.4

Runge–Kutta Methods

Your turn: compare equal model-call budgets

Solution 3 / 4

3 · Verify the resulting Euler run

h: physical interval of one RK4 step; h=0.1 · nE,hE: Euler substep count and substep size · model call: one evaluation of f(x)=−2x · x4,E: state after four Euler substeps

1. For this scalar model, each substep multiplies the state by 0.95.

$$x_{4,E} = (1 - 2h_E)^4 x_0 = (1 - 0.05)^4 (1) = 0.95^4 = 0.81450625.$$

Function evaluations; order of accuracy — [4] FNC §6.4

EXERCISE · SLIDE 38 · SOLUTION 4/4

Runge–Kutta Methods

Your turn: compare equal model-call budgets

Solution 4 / 4

Answer · A

h: physical interval of one RK4 step; h=0.1 · nE,hE: Euler substep count and substep size · model call: one evaluation of f(x)=−2x · x4,E: state after four Euler substeps

1. A matches both cost and interval.
2. B matches the interval but uses only one call.
3. C matches cost but advances four times as far.

Function evaluations; order of accuracy — [4] FNC §6.4

EXERCISE · SLIDE 38 · CHECK THE RESULT

Runge–Kutta Methods

Your turn: compare equal model-call budgets

$$\dot{x} = -2x, \quad x_0 = 1, \quad h = 0.1$$

`h` one RK4 step interval

`cost` number of calls to f

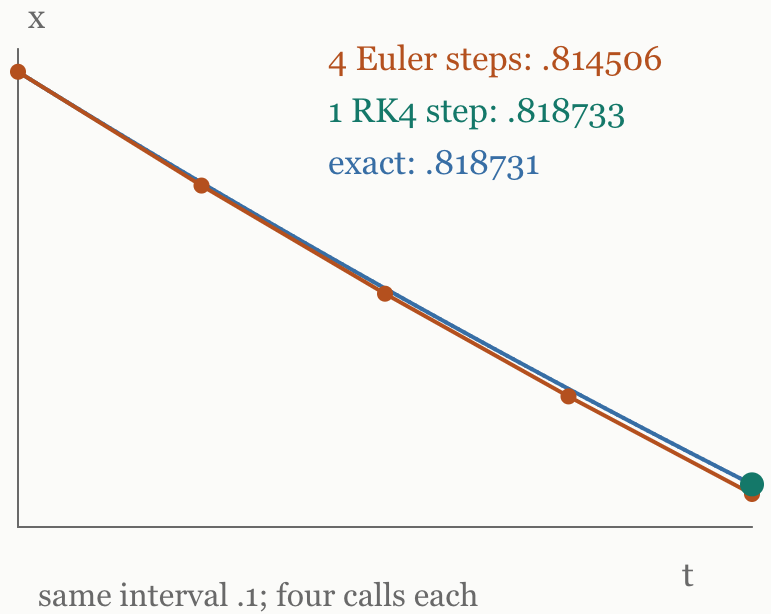
How many Euler substeps match ONE RK4 step in model calls?

A. Four, each h/4, over the same interval

B. One, with step h

C. Four, each h, over the same interval

> ✓
>
> Correct A. Match cost AND interval.



Function evaluations; order of accuracy — [4] FNC §6.4

EXERCISE · SLIDE 43 · QUESTION

Integrated Pendulum Problem

Choose the state and the check first

$$\ddot{q} = -\sin q, \quad q_0 = 1, \quad v_0 = 0$$

`q₀,v₀` angle and angular velocity

`model` unforced, undamped; m=ℓ=g0=1

Which state and field give a first-order ODE for this model?

A. x=q; f=−sin q

B. x=(q,v)ᵀ; f=(v,−sin q)ᵀ

C. x=(q,v)ᵀ; f=(−sin q,v)ᵀ

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 43 · SOLUTION 1/4

Integrated Pendulum Problem

Choose the state and the check first

Solution 1 / 4

1 · Introduce velocity

q,v: angle and angular velocity · x: complete column state · f(x): state derivative for normalized undamped dynamics

1. A second-order equation needs both position and velocity to determine the future.

$$v = \dot{q}, \qquad \ddot{q} = -\sin q \quad \Longrightarrow \quad \dot{v} = -\sin q.$$

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Choose the state and the check first

Solution 2 / 4

2 · Stack the state and its rates

q,v: angle and angular velocity · x: complete column state · f(x): state derivative for normalized undamped dynamics

1. Use the same coordinate order in x and f.

$$x = \begin{bmatrix} q \\ v \end{bmatrix},$$

$$\dot{x} = f(x) = \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\sin q \end{bmatrix}.$$

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Choose the state and the check first

Solution 3 / 4

3 · Apply the given initial state

q,v: angle and angular velocity · x: complete column state · f(x): state derivative for normalized undamped dynamics

1. The pendulum starts at angle one with zero angular velocity.

$$x_0 = (1, 0)^T, \qquad f(x_0) = (0, -\sin 1)^T \approx (0, -0.84147098)^T.$$

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 43 · SOLUTION 4/4

Integrated Pendulum Problem

Choose the state and the check first

Solution 4 / 4

Answer · B

q,v: angle and angular velocity · x: complete column state · f(x): state derivative for normalized undamped dynamics

1. A omits velocity and mistakes acceleration for angle rate.
2. B carries both state coordinates and returns their rates in order.
3. C swaps the rate components.

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 43 · CHECK THE RESULT

Integrated Pendulum Problem

Choose the state and the check first

$$\ddot{q} = -\sin q, \quad q_0 = 1, \quad v_0 = 0$$

`q₀,v₀` angle and angular velocity

`model` unforced, undamped; m=ℓ=g0=1

Which state and field give a first-order ODE for this model?

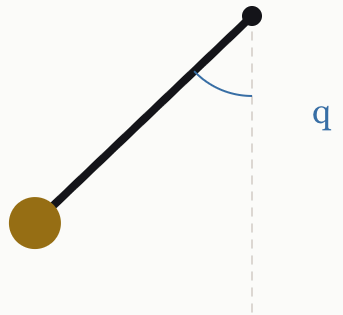
A. x=q; f=−sin q

B. x=(q,v)ᵀ; f=(v,−sin q)ᵀ

C. x=(q,v)ᵀ; f=(−sin q,v)ᵀ

> ✓
>
> Correct B. Keep state/rate order.

Schematic angle; not an integrated trajectory



downward reference

Pendulum IVP; numerical integration — [1] Tedrake, ch.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 44 · QUESTION

Integrated Pendulum Problem

Compare states at t = 0.1

$$x_0 = (1, 0)^T,\ h = .1,\ f(q, v) = (v, -\sin q)^T$$

`x=(q,v)ᵀ` angle and angular velocity

`h` full time step

Why can midpoint give $q_1 \neq q_0$ even though $v_0 = 0$?

A. Nonzero velocity at the trial midpoint.

B. Twice the physical time.

C. A different initial velocity.

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 1 / 12

1 · Evaluate the initial rate

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. The initial angle rate is zero, but the acceleration is nonzero.

$$k_1 = f(x_0) = (0, -\sin 1)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 2 / 12

2 · Form the trial state

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. The half-step changes velocity even though the trial angle remains one.

$$x_{\text{mid}} = x_0 + \tfrac{h}{2} k_1 = (1, -0.05 \sin 1)^T \approx (1, -0.04207355)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 3 / 12

3 · Evaluate the trial rate

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. The trial velocity becomes the nonzero first component of the new rate.

$$k_2 = f(x_{\text{mid}}) = (-0.05 \sin 1, -\sin 1)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 4 / 12

4 · Compute the next state with midpoint

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. Start from x0 and advance one full step using k2.

$$x_1 = x_0 + 0.1k_2 = (1 - 0.005\sin 1, -0.1\sin 1)^T \approx (0.99579265, -0.08414710)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 5 / 12

5 · Compare with Euler

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. Euler uses the zero initial velocity for the angle update; midpoint uses the nonzero trial velocity.

$$q_{1,E} = 1 + 0.1(0) = 1, \qquad q_{1,\text{mid}} = 1 - 0.005 \sin 1 < 1.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 6 / 12

6 · RK4 first two stages

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. For RK4, define a new set of stage rates k1,…,k4. Each z is an internal trial state.

$$k_1 \approx (0, -0.8414709848)^T, \quad z_2 = x_0 + 0.05k_1 \approx (1, -0.04207354924)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 7 / 12

7 · RK4 second and third rates

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. Evaluate the same field at each trial state.

$$k_2 = f(z_2) \approx (-0.04207354924, -0.8414709848)^T. \quad z_3 = x_0 + 0.05k_2 \approx (0.9978963225, -0.04207354924)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 8 / 12

8 · RK4 final trial

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. Use the third rate to build the full-step trial.

$$k_3 = f(z_3) \approx (-0.04207354924, -0.8403325019)^T, \quad z_4 = x_0 + 0.1k_3 \approx (0.9957926451, -0.08403325019)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 9 / 12

9 · RK4 final rate

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. The fourth rate is evaluated at the full-step trial, not the accepted state.

$$k_4 = f(z_4) \approx (-0.08403325019, -0.8391903002)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 10 / 12

10 · Weight the four rates

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. The two midpoint rates each receive weight two.

$$k_1 + 2k_2 + 2k_3 + k_4 \approx (-0.2523274472, -5.044268259)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 11 / 12

11 · Accept the RK4 state

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. Scale the weighted rate sum by h/6, then add it to the original state.

$$x_{1,\mathrm{RK4}} = x_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \approx (0.9957945425, -0.08407113764)^T.$$

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Compare states at t = 0.1

Solution 12 / 12

Answer · A

q,v: angle and angular velocity · x0,h: initial state $(1,0)^T$ and full step 0.1 · k_i: model rate at stage i · xmid,z_i: midpoint and RK4 trial states

1. A uses acceleration to form a trial velocity and hence a nonzero angle rate.
2. B mistakes an internal trial for an extra time interval.
3. C changes the data; all methods share the same start.

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 44 · CHECK THE RESULT

Integrated Pendulum Problem

Compare states at t = 0.1

$$x_0 = (1, 0)^T,\ h = .1,\ f(q, v) = (v, -\sin q)^T$$

`x=(q,v)ᵀ` angle and angular velocity

`h` full time step

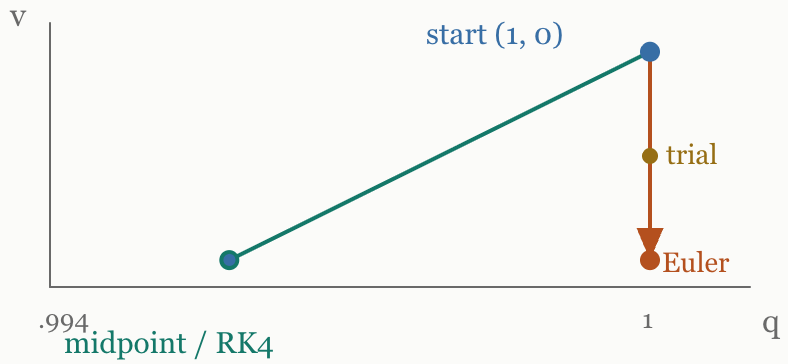
Why can midpoint give $q_1 \neq q_0$ even though $v_0 = 0$?

A. Nonzero velocity at the trial midpoint.

B. Twice the physical time.

C. A different initial velocity.

> ✓
>
> Correct A. Trial velocity need not be zero.



METHOD	Q_1	V_1
Euler	1.000000	−.084147
midpoint	.995793	−.084147
RK4	.995795	−.084071

Euler and explicit midpoint methods — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 45 · QUESTION

Integrated Pendulum Problem

Repeat to the same final time

$$\dot{x} = (v, -\sin q)^T, \quad x(0) = (1, 0)^T, \quad T = 10$$

N	step count
E	$v^2/2+1-\cos q$
ΔE	E at T minus E at 0

Halve h: how do we still end at time T=10?

A. Keep N.

B. Stop at equal E.

C. Double N.

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 45 · SOLUTION 1/8

Integrated Pendulum Problem

Repeat to the same final time

Solution 1 / 8

1 · Express the endpoint

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. For constant steps, final time is step count times step size.

$$T = Nh \quad \Longrightarrow \quad N = T/h.$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 2 / 8

2 · Halve the step

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. Keeping T fixed forces the new step count to double.

$$h_{\text{new}} = h/2, \qquad N_{\text{new}} = \frac{T}{h/2} = 2\frac{T}{h} = 2N.$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 3 / 8

3 · Check with numbers

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. All three runs end at the same physical time.

h	0.2	0.1	0.05
$N = 10/h$	50	100	200

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 4 / 8

4 · Separate the comparison from its diagnostic

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. Compute energy error at the prescribed endpoint; stopping at equal energy would compare different times.

$$\Delta E_N = E(x_N) - E(x_0), \qquad t_N = T.$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 5 / 8

5 · Reproduce the Euler column

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. Start at $x_0=(1,0)^T$. For $k=0,\dots,N-1$, use both components from the old state.

$$q_{k+1} = q_k + h v_k, \qquad v_{k+1} = v_k - h \sin q_k.$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 6 / 8

6 · Reproduce the RK4 column

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. At every step, evaluate $f(q,v)=(v,-\sin q)^T$ at the four stage states.

$$k_1 = f(x_k), \quad k_2 = f(x_k + hk_1/2), \quad k_3 = f(x_k + hk_2/2), \quad k_4 = f(x_k + hk_3).$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

Integrated Pendulum Problem

Repeat to the same final time

Solution 7 / 8

7 · Finish each run and evaluate the diagnostic

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. Repeat the update N=T/h times. The table reports this numerical iteration, not a closed-form trajectory.

$$x_{k+1} = x_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad \Delta E_N = \frac{v_N^2}{2} + 1 - \cos q_N - (1 - \cos 1).$$

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 45 · SOLUTION 8/8

Integrated Pendulum Problem

Repeat to the same final time

Solution 8 / 8

Answer · C

T: common final time, 10 · h,N: step size and integer step count · E,ΔE: energy and its change · k_i: RK4 stage rates; $x=(q,v)^T$

1. A halves the endpoint time.
2. B replaces the prescribed time with a diagnostic-dependent endpoint.
3. C preserves the physical final time.

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 45 · CHECK THE RESULT

Integrated Pendulum Problem

Repeat to the same final time

$$\dot{x} = (v, -\sin q)^T, \quad x(0) = (1, 0)^T, \quad T = 10$$

- **N** step count
- **E** $v^2/2+1-\cos q$
- **ΔE** E at T minus E at 0

Halve h: how do we still end at time T=10?

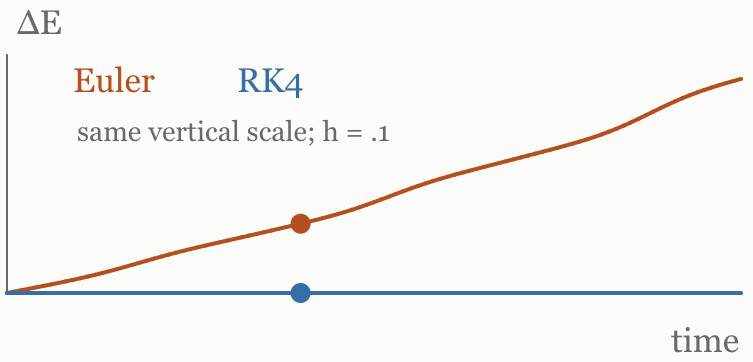
A. Keep N.

B. Stop at equal E.

C. Double N.

✓

Correct C: keep Nh=T.



H	EULER ΔE	RK4 ΔE
.2	1.324213	-1.46×10^{-5}
.1	.539746	-4.61×10^{-7}
.05	.232375	-1.44×10^{-8}

Global error; step refinement — [4] FNC §6.2 · [4] FNC §6.4 · Course example

EXERCISE · SLIDE 46 · QUESTION

Integrated Pendulum Problem

Compare two energy-matched states

$$E(q, v) = v^2/2 + 1 - \cos q$$

`q,v` angle and angular velocity

`E` energy

What does equal energy of two pendulum states establish?

A. Their angles and velocities are equal.

B. They can still be different states.

C. Both numerical trajectories are exact.

Energy level; state error — [1] Tedrake, ch.2 · Course example

EXERCISE · SLIDE 46 · SOLUTION 1/5

Integrated Pendulum Problem

Compare two energy-matched states

Solution 1 / 5

1 · Choose different states

q,v: angle and angular velocity · E: energy $v^2/2+1-\cos q$ · xa,xb: two test states · x_k,$x(t_k)$: numerical and exact states at time t_k

1. Use opposite angles and zero velocities.

$$x_a = (1, 0)^T, \qquad x_b = (-1, 0)^T, \qquad x_a \neq x_b.$$

Energy level; state error — [1] Tedrake, ch.2 · Course example

Integrated Pendulum Problem

Compare two energy-matched states

Solution 2 / 5

2 · Evaluate both energies

q,v: angle and angular velocity · E: energy $v^2/2+1-\cos q$ · xa,xb: two test states · x_k,$x(t_k)$: numerical and exact states at time t_k

1. Cosine is even, so opposite angles give identical potential energy.

$$E(x_a) = 1 - \cos 1, \qquad E(x_b) = 1 - \cos(-1) = 1 - \cos 1.$$

Energy level; state error — [1] Tedrake, ch.2 · Course example

Integrated Pendulum Problem

Compare two energy-matched states

Solution 3 / 5

3 · Test the claimed implication

q,v: angle and angular velocity · E: energy $v^2/2+1-\cos q$ · xa,xb: two test states · x_k,$x(t_k)$: numerical and exact states at time t_k

1. One counterexample disproves the claim that equal energy forces equal states.

$$E(x_a) = E(x_b) \quad \text{but} \quad \|x_a - x_b\|_2 = \|(2,0)^T\|_2 = 2.$$

Energy level; state error — [1] Tedrake, ch.2 · Course example

Integrated Pendulum Problem

Compare two energy-matched states

Solution 4 / 5

4 · State what the check can establish

q,v: angle and angular velocity · E: energy $v^2/2+1-\cos q$ · xa,xb: two test states · x_k,$x(t_k)$: numerical and exact states at time t_k

1. Energy is one useful diagnostic. State accuracy also needs comparison at common physical times.

$$E(x_k) \approx E(x_0) \;\not\Rightarrow\; x_k \approx x(t_k).$$

Energy level; state error — [1] Tedrake, ch.2 · Course example

EXERCISE · SLIDE 46 · SOLUTION 5/5

Integrated Pendulum Problem

Compare two energy-matched states

Solution 5 / 5

Answer · B

q,v: angle and angular velocity · E: energy $v^2/2+1-\cos q$ · xa,xb: two test states · x_k,$x(t_k)$: numerical and exact states at time t_k

1. A treats one scalar as a complete state.
2. B allows opposite angles with zero velocity.
3. C mistakes one invariant check for proof of accuracy.

Energy level; state error — [1] Tedrake, ch.2 · Course example

EXERCISE · SLIDE 46 · CHECK THE RESULT

Integrated Pendulum Problem

Compare two energy-matched states

$$E(q, v) = v^2/2 + 1 - \cos q$$

`q,v` angle and angular velocity

`E` energy

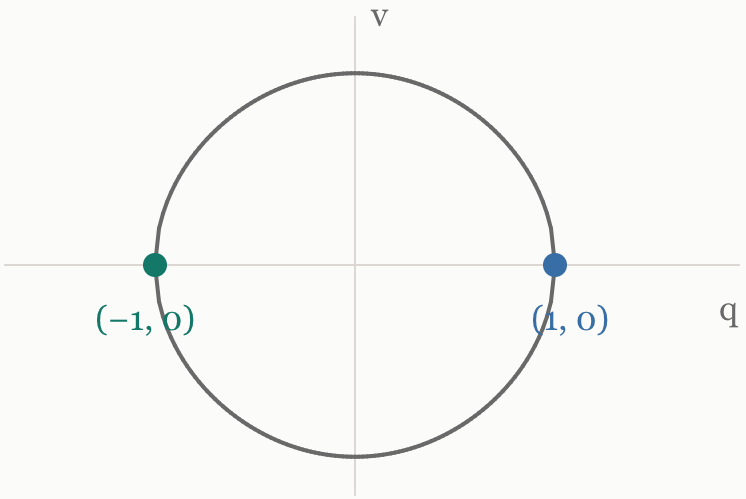
What does equal energy of two pendulum states establish?

A. Their angles and velocities are equal.

B. They can still be different states.

C. Both numerical trajectories are exact.

> ✓
>
> Correct B. Energy cannot fix phase.



E = 1 − cos 1 at both states

Energy level; state error — [1] Tedrake, ch.2 · Course example

EXERCISE · SLIDE 47 · QUESTION

Independent Transfer

Transfer: a moving pendulum near the top

$$q_0 = \pi, \quad v_0 = 1, \quad h = .1, \quad m = \ell = g_0 = 1$$

`q₀,v₀`	initial angle and angular velocity
`f(q,v)`	$(v, -\sin q)^T$
`E(q,v)`	$v^2/2 + 1 - \cos q$

One call: which method and energy claim is correct?

A. Euler; this step decreases energy.

B. Midpoint; one call is enough.

C. Euler; every step increases energy.

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

EXERCISE · SLIDE 47 · SOLUTION 1/6

Independent Transfer

Transfer: a moving pendulum near the top

Solution 1 / 6

1 · Select the affordable method

q_0, v_0: initial angle π and angular velocity 1 · h: step size 0.1 · $E, \Delta E$: total energy and its change · x_0, x_1: initial and accepted state

1. Euler uses one model call; midpoint uses two; RK4 uses four. Only Euler fits this budget.

$$1 \leq 1, \qquad 2 > 1, \qquad 4 > 1.$$

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

Independent Transfer

Transfer: a moving pendulum near the top

Solution 2 / 6

2 · Evaluate the rate and update

q0,v0: initial angle π and angular velocity 1 · h: step size 0.1 · E,ΔE: total energy and its change · x0,x1: initial and accepted state

1. At the top, sine is zero, so the instantaneous acceleration is zero.

$$f(x_0) = (1, -\sin\pi)^T = (1, 0)^T, \qquad x_1 = x_0 + 0.1 f(x_0) = (\pi + 0.1, 1)^T.$$

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

Independent Transfer

Transfer: a moving pendulum near the top

Solution 3 / 6

3 · Compute the initial energy

q_0,v_0: initial angle π and angular velocity 1 · h: step size 0.1 · E,ΔE: total energy and its change · x_0,x_1: initial and accepted state

1. Kinetic energy is one half; potential energy is two.

$$E_0 = \tfrac{1}{2}(1)^2 + 1 - \cos\pi = \tfrac{1}{2} + 2 = \tfrac{5}{2}.$$

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

EXERCISE · SLIDE 47 · SOLUTION 4/6

Independent Transfer

Transfer: a moving pendulum near the top

Solution 4 / 6

4 · Compute the new energy

q_0,v_0: initial angle π and angular velocity 1 · h: step size 0.1 · E,ΔE: total energy and its change · x_0,x_1: initial and accepted state

1. The velocity is unchanged; use $\cos(\pi+a)=-\cos(a)$ for the potential term.

$$E_1 = \tfrac{1}{2} + 1 - \cos(\pi + 0.1) = \tfrac{3}{2} + \cos(0.1).$$

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

Independent Transfer

Transfer: a moving pendulum near the top

Solution 5 / 6

5 · Subtract and determine the sign

q_0,v_0: initial angle π and angular velocity 1 · h: step size 0.1 · E,ΔE: total energy and its change · x_0,x_1: initial and accepted state

1. Since $0<0.1<2\pi$, cosine is strictly below one. This particular Euler step loses energy.

$$\Delta E = E_1 - E_0 = \cos(0.1) - 1 \approx -0.004995835 < 0.$$

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

EXERCISE · SLIDE 47 · SOLUTION 6/6

Independent Transfer

Transfer: a moving pendulum near the top

Solution 6 / 6

Answer · A

q_0, v_0: initial angle π and angular velocity 1 · h: step size 0.1 · $E, \Delta E$: total energy and its change · x_0, x_1: initial and accepted state

1. A uses Euler and the potential drops while kinetic energy stays fixed.
2. B requires two calls, exceeding this budget.
3. C incorrectly generalizes the earlier zero-velocity example.

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

EXERCISE · SLIDE 47 · CHECK THE RESULT

Independent Transfer

Transfer: a moving pendulum near the top

$$q_0 = \pi, \quad v_0 = 1, \quad h = .1, \quad m = \ell = g_0 = 1$$

`q₀,v₀`	initial angle and angular velocity
`f(q,v)`	$(v, -\sin q)^T$
`E(q,v)`	$v^2/2 + 1 - \cos q$

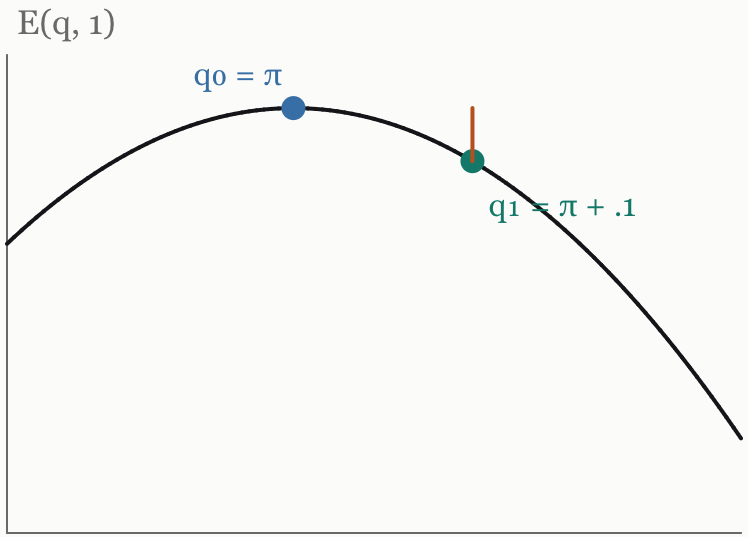
One call: which method and energy claim is correct?

A. Euler; this step decreases energy.

B. Midpoint; one call is enough.

C. Euler; every step increases energy.

> ✓
>
> Correct A. Potential falls; kinetic energy is unchanged.



$E_1 < E_0$; $v = 1$ at both states

Pendulum IVP; explicit Euler method — [1] Tedrake, ch.2 · [4] FNC §6.2 · Course example

EXERCISE · SLIDE 48 · QUESTION

Simulation Checklist

Simulation workflow

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0$$

`x`	complete state
`model`	unforced, undamped pendulum; $\ddot{q} = -\sin q$
`h`	chosen integration step

For this pendulum simulation, which workflow distinguishes errors?

A. Use q alone; shrink h until the curve looks smooth.

B. Use f as the next state; inspect only energy.

C. Use (q,v), integrate rates, compare states and energy.

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

EXERCISE · SLIDE 48 · SOLUTION 1/5

Simulation Checklist

Simulation workflow

Solution 1 / 5

1 · Choose a complete state

$x=(q,v)^T$: complete pendulum state · f,h: model rate and numerical step size · E: energy used alongside state comparisons · k,t_k: step index and time $t_k=kh$

1. Angle alone omits velocity; the same angle can lead to different future motion.

$$x = (q, v)^T, \qquad f(x) = (v, -\sin q)^T.$$

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

Simulation Checklist

Simulation workflow

Solution 2 / 5

2 · Integrate the rate

$x=(q,v)^T$: complete pendulum state · f,h: model rate and numerical step size · E: energy used alongside state comparisons · k,t_k: step index and time $t_k=kh$

1. For example, Euler integrates the current rate over one step: multiply the rate by h, then add it to the old state.

$$x_{k+1} = x_k + hf(x_k, t_k).$$

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

EXERCISE · SLIDE 48 · SOLUTION 3/5

Simulation Checklist

Simulation workflow

Solution 3 / 5

3 · Compare states at common times

$x=(q,v)^T$: complete pendulum state · f,h: model rate and numerical step size · E: energy used alongside state comparisons · k,t_k: step index and time $t_k=kh$

1. Reduce the step and compare against an exact solution or a suitably refined calculation. A smooth curve alone is not an accuracy test.

$$t_k = kh = 2k(h/2).$$

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

Simulation Checklist

Simulation workflow

Solution 4 / 5

4 · Use a complementary invariant check

$x=(q,v)^T$: complete pendulum state · f,h: model rate and numerical step size · E: energy used alongside state comparisons · k,t_k: step index and time $t_k=kh$

1. For this undamped, unforced model, exact energy stays constant. Also inspect the full state: equal energy need not imply equal state.

$$E(q,v) = v^2/2 + 1 - \cos q, \qquad \Delta E_k = E(x_k) - E(x_0).$$

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

EXERCISE · SLIDE 48 · SOLUTION 5/5

Simulation Checklist

Simulation workflow

Solution 5 / 5

Answer · C

$x=(q,v)^T$: complete pendulum state · f,h: model rate and numerical step size · E: energy used alongside state comparisons · k,t_k: step index and time $t_k=kh$

1. A omits velocity, which a smaller step cannot repair.
2. B confuses rates with states and energy with full accuracy.
3. C separates representation, update and complementary diagnostics.

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist

EXERCISE · SLIDE 48 · CHECK THE RESULT

Simulation Checklist

Simulation workflow

$$\dot{x} = f(x,t), \qquad x(t_0) = x_0$$

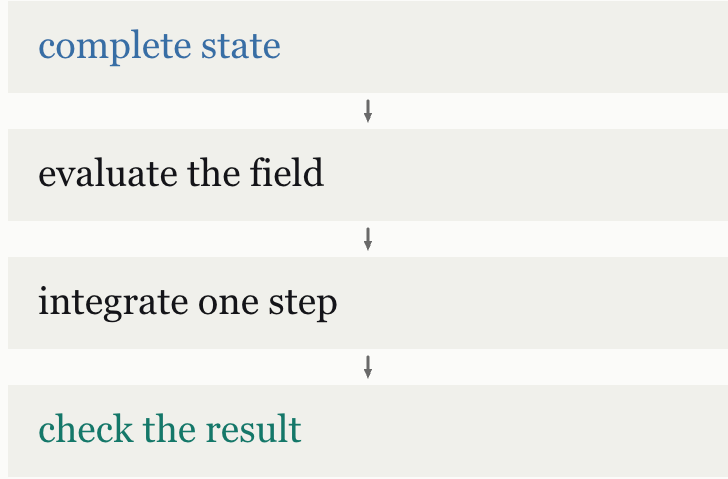
`x`	complete state
`model`	unforced, undamped pendulum; $\ddot{q} = -\sin q$
`h`	chosen integration step

For this pendulum simulation, which workflow distinguishes errors?

A. Use q alone; shrink h until the curve looks smooth.

B. Use f as the next state; inspect only energy.

C. Use (q,v), integrate rates, compare states and energy.



✓

Correct C. Check state AND invariant.

Verification; validation; step refinement — [5] NASA, V&V glossary · [4] FNC §6.2 · Course checklist